Manonmaniam Sundaranar University,
Directorate of Distance & Continuing Education,
Tirunelveli - 627 012 Tamilnadu, India

OPEN AND DISTANCE LEARNING (ODL) PROGRAMMES
(FOR THOSE WHO JOINED THE PROGRAMMES FROM THE ACADEMIC YEAR 2023–2024)

II YEAR

M.Sc. Physics

Course Material

Numerical Methods and Programming in C++

Prepared
By

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Tirunelveli – 12

NUMERICAL METHODS AND PROGRAMMING In C++

Unit –I

Roots of Equation:

Roots of equation: Bisection method – False position method – Newton Raphson method – Secant method – Order of convergence Simultaneous Equation Existence of solutions – Basic Gauss elimination method – Gauss Jacobi iteration method – Gauss Seidal iteration method – Inverse of a matrix using Gauss elimination method.

Unit –II

Curve Fitting – Interpolation:

Curve fitting: Method of least squares – straight line, fitting a parabola, fitting y= axn, y = aebx type curves – Interpolation: Polynomial Interpolation – Lagrange polynomial – Newton polynomial – Forward and Backward differences – Gregory Newton forward and backward interpolation formula for equal intervals – Divided difference – properties of divided differences – Newton's divided differences formula – Lagrange's interpolation formula for unequal interval.

Unit –III

Eigen Values, Differentiation and Integration:

Eigenvalues: Power method to find dominant Eigenvalue – Jacobi method

Numerical differentiation: Numerical differentiation – Formulae for derivatives – Taylors Series Method – Forward backward differences and central difference formula

Numerical Integration: Newton – cotes formula – Trapezoidal rule, Simpon's 1/3 rule, Simpson's 3/8 rule, - Error estimates in trapezoidal and Simpson's rule – Monte Carlo Method.

Unit –IV

Differential Equations

Ordinary differential equation: Solution by Taylor's series – Basic Euler method – Improved and modified Euler method – Runge Kutta fourth order method – solution of simultaneous first order differential equations and second order differential equations by RK fourth order Method

Partial differential equation:

Introduction – Classification of partial differential equation of the 2^{nd} order – Finite Difference approximations – Solution of Laplace's equation – Solution of Poisson's Equation – standard five point formula and diagonal five point formula (Jacobi and Gauss Seidal Methods).

Unit –V

Programming In C++:

Program structure and header files – Basic data types – operators – Control Structure: decision making and looping statements. Arrays, Strings, Structures, Pointers and File handling. Application programs – Solution to Algebraic and transcendental equations by Newton Raphson Method – Charging and discharging of a condenser by Euler's Method – Radioactive Decay by Runge Kutta fourth order method – Currents in Wheatstone's bridge by Gauss elimination method – Cauchy's constant by least square method – Evaluation of integral by Simpson's and Monte – Carlo methods – Newton's Law of cooling by Numerical differentiation.

Unit –VI

Professional Components:

Expert Lectures, Online Seminars – Webinars on Industrial Interactions/Visits, Competitive Examinations, Employable and Communication Skill Enhancement, Social Accountability and Patriotism.

TEXT BOOKS

Introductory methods of numerical analysis, S.S. Sastry, Prentice Hall of India, 2010

Numerical methods for mathematics, science and engineering, John H. Matthews, Prentice Hall of India, 2nd Edition, 2000

M.K. Jain, S.R.K. Iyengar, R. K. Jain, Numerical Methods for Scientific and Engineering computation, 3rd edition, New age international (P) Ltd, Chennai, 1998.

Object Oriented Programming with C++ by E. Balagurusamy, Tata McGraw- Hill, India, 4th Edition

Unit –I

Roots of Equation:

Roots of equation: Bisection method – False position method – Newton Raphson method – Secant method – Order of convergence Simultaneous Equation Existence of solutions – Basic Gauss elimination method – Gauss Jacobi iteration method – Gauss Seidal iteration method – Inverse of a matrix using Gauss elimination method.

1.1. Roots of equation:

Algebraic Equations:

The equations involving polynomials in x are called algebraic equations.

(eg). $x^{-1} + 5x + 10 = 0$

Numeric algebraic equations:

If the coefficients of the polynomial are pure numbers then the equations are called numeric algebraic equations.

(eg). 2x

Transcendental equations:

Equations involving transcendental functions are called as transcendental equations.

(eg). Sinx, Cosx

Numeric transcendental equations:

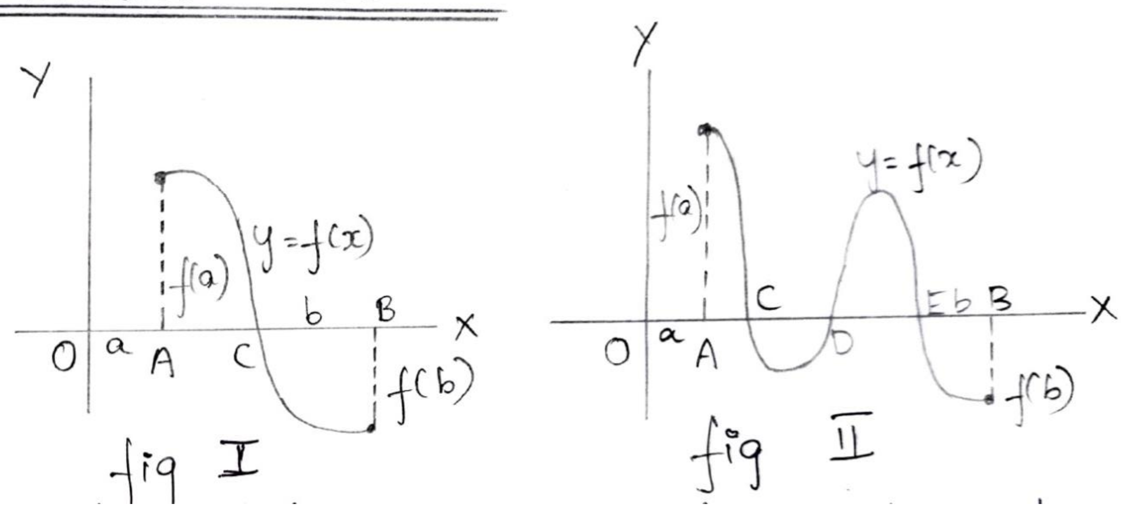
If the coefficient of transcendental equations are real then the equations is known as numeric transcendental equations.

(eg) $7e^{x} - Sin\, x + 5\log x + 1 = 0$

Fundamental Theorem:

The fundamental theorem will enable us to locate the real root of an equation f(x) = 0. If f(x) is continuous from x=a and x=b if f (a) and f (b) are of opposite signs, then the equations F(x) = 0 will have at least one root between a and b.

Fundamental Theorem:



Graphical consideration:

If we draw the graph of the continuous function y= f(x) we find that if f(a) and f(b) are of opposite signs, then the graph must cut the x axis at least once as shown in figure 1. And always at odd no. of. Times as shown in figure II. Hence at C, there is a root of the equation f(x) = 0 between a and b.

1.2. Bisection Method:

Let the function f(x) be continuous between a and b.

Let f (a) be –ve and f(b) be +ve.

Then there is a root of f(x) = 0 lying between a and b.

The first approximate value is given by $x_0 = \frac{a+b}{2}$.

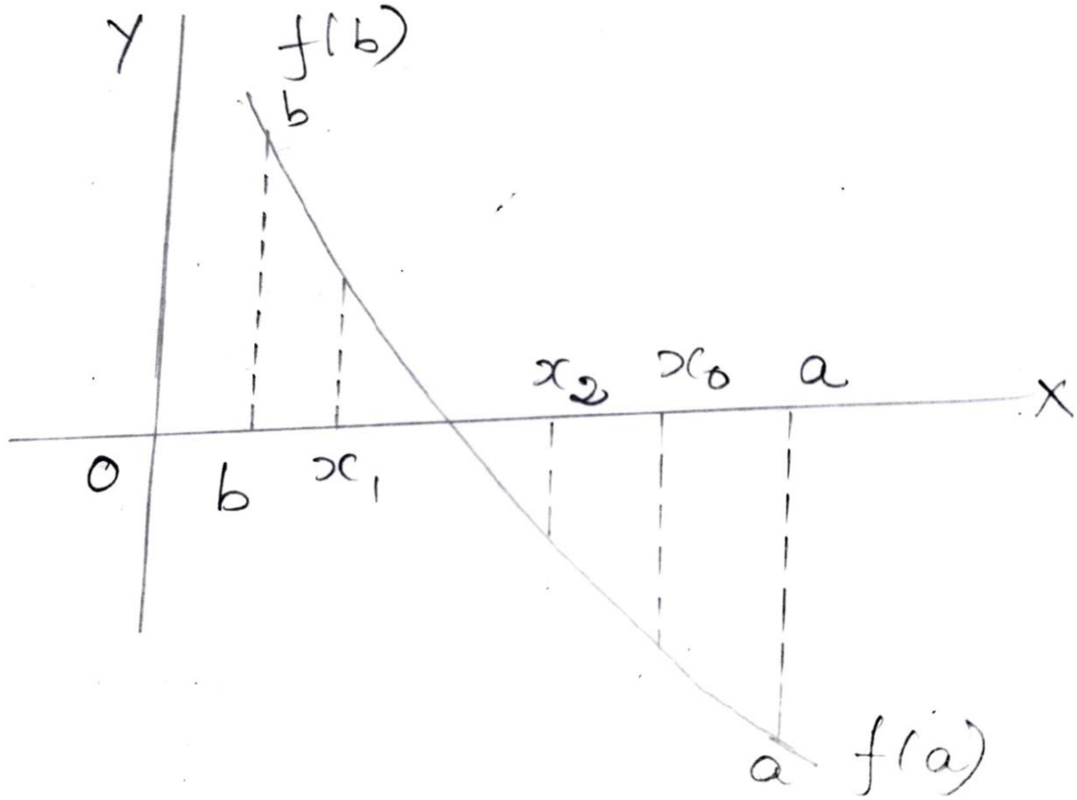
If f (x_0) = 0, it means that x_0 *is a root of* $f(x) = 0$

Otherwise the root lies between x_0 *and* b *or* x_0 *and* a according to f (x_0) is positive or negative.

If f (x_0) = +ve the root lies between x_0 and a.

If f (x_0) = -ve the root lies between x_0 and b.

Therefore we bisect the interval and continue the process until the root is known to desired accuracy.



In the figure first approximation, $x_0 = \frac{a+b}{2}$

and f (x_0) = -ve the root lies between x_0 and b.

Second approximation, $x_1 = \frac{x_0+b}{2}$. Suppose f (x_1) is positive,

Third approximation, $x_2 = \frac{x_1+x_0}{2}$ and so on.

Bisection method is simple but slowly convergent method.

Problem 1:

Find a root of the equation $x^3 - 4x - 9 = 0$ correct to 3 decimal places by bisection method.

Solution :

f (x) = $x^3 - 4x - 9 = 0$

f (x) = -9 = -ve

f (x) = 1 – 4 – 9 = -12 = -ve

f (x) = 8 – 8 – 9 = -9 = -ve

f (x) = 27 – 12 – 9 = 6 = +ve

Therefore a root lies between 2 and 3

$$x_0 = \frac{2+3}{2} = 2.5$$

f (2.5) = -3.375 = -ve

f (x_0) = -ve

The root lies between x_0 and 3

$$x_1 = \frac{2.5+3}{2} = 2.75$$

f (x_1) = 0.796875 = +ve

The root lies between x_1 and x_0

$$x_2 = \frac{x_1 + x_0}{2} = \frac{2.75+2.5}{2}$$

$$x_2 = 2.625$$

f (x_2) = -1.4121 = -ve

The root lies between x_2 and x_1

$$x_3 = \frac{x_1 + x_2}{2} = \frac{2.625+2.75}{2}$$

$$x_3 = 2.6875$$

f (x_3) = -0.3391 = -ve

Root lies between x_3 and x_1

$$x_4 = \frac{x_1 + x_3}{2} = \frac{2.75+2.6875}{2}$$

$$x_4 = 2.71875$$

f (x_4) = 0.220917 = +ve

Root lies between x_4 and x_3

$$x_5 = \frac{x_3 + x_4}{2} = \frac{2.6875+2.71875}{2}$$

$$x_5 = 2.703125$$

f (x_5) = -0.06107 = -ve

Root lies between x_5 and x_4

$$x_6 = \frac{x_4 + x_5}{2} = \frac{2.71875+2.703125}{2}$$

$$x_6 = 2.7109375$$

f (x_6) = 0.07942 = +ve

Root lies between x_6 and x_5

$$x_7 = \frac{x_5 + x_6}{2} = \frac{2.703125 + 2.7109375}{2}$$

$$x_7 = 2.70703125$$

$$f(x_7) = 9.04923 \times 10^{-3} = +ve$$

Root lies between x_7 and x_5

$$x_8 = \frac{x_7 + x_5}{2} = \frac{2.70703125 + 2.703125}{2}$$

$$x_8 = 2.705078125$$

$$f(x_8) = -0.02604 = -ve$$

Root lies between x_8 and x_7

$$x_9 = \frac{2.705078125 + 2.70703125}{2}$$

$$x_9 = 2.706054688$$

$$f(x_9) = -8.5055 \times 10^{-3} = -ve$$

Root lies between x_9 and x_7

$$x_{10} = \frac{2.70703125 + 2.706054688}{2}$$

$$x_{10} = 2.706542929$$

$$f(x_{10}) = 2.6990 \times 10^{-4} = +ve$$

Root lies between x_{10} and x_7

$$x_{11} = \frac{2.70703125 + 2.706542969}{2}$$

$$x_{11} = 2.70678711$$

We found that the value of the root has settled down to three places.

Hence the root is 2.706.

Problem 2

Obtain a root of the equations correct to 3 decimal places by using Bisection method

$$x^3 - x - 1 = 0.$$

Solution: 1.324

Problem 3

Obtain a root of the following equations correct to three decimal places by bisection method

$$x^3 - 9x + 1 = 0.$$

Problem 4

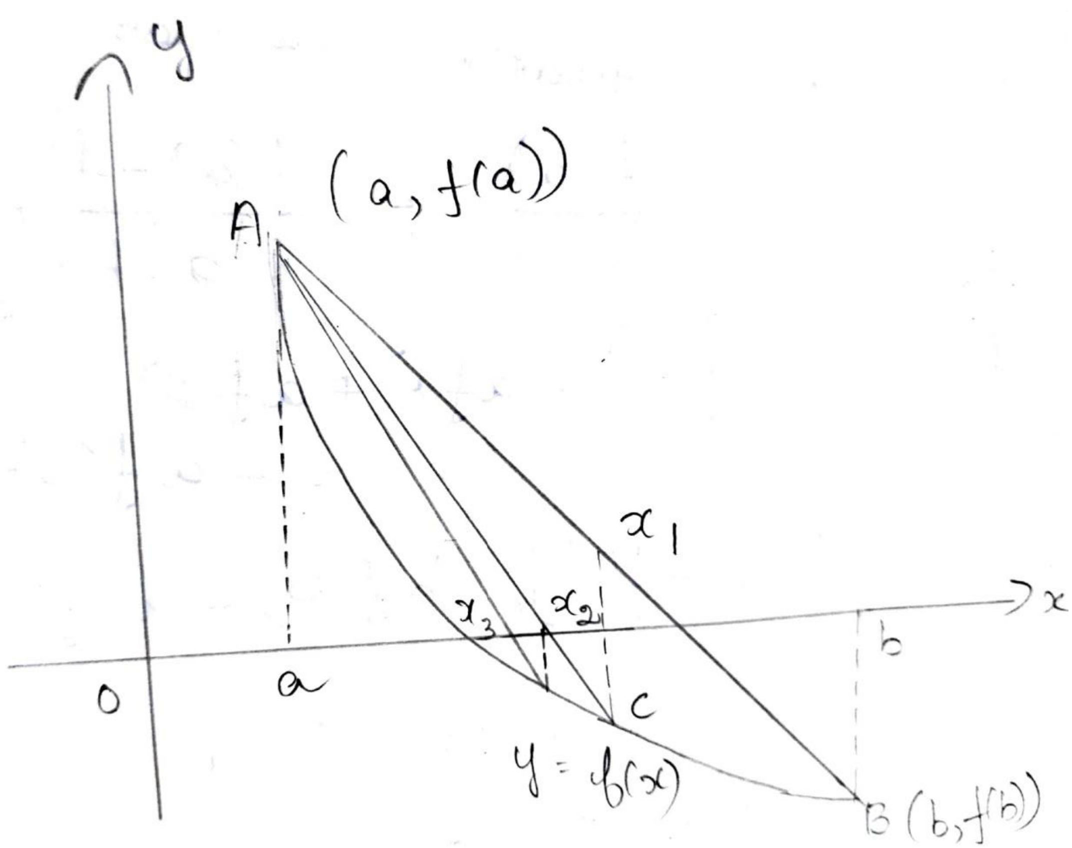
Obtain a root of the equation correct to 3 decimal places by using bisection method

$$x^3 - x^2 + x - 7 = 0.$$

Solution: 2.104

1.3. Regula False Method:

Consider the equation f(x) = 0 and let f(a) and f(b) be of opposite signs.



The curve y = f(x) will meet the x axis at some point between A(a, f(a)) and B(b, f(b)).

The equation of chord joining the two points A(a, f(a)) and B(b, f(b)) is

$$\frac{y-f(a)}{x-a} = \frac{f(a)-f(b)}{a-b}$$

The x coordinate of the point of intersection of this chord with the x-axis gives an approximate value for the root of f(x) = 0. Setting y = 0 in chord equation. We get

$$\frac{-f(a)}{x-a} = \frac{f(a)-f(b)}{a-b}$$

$$X\,(f(a)-f(b)) – af(a) + af(b) = -af(a) + bf(a)$$

$$X\,(f(a)-f(b)) = bf(a) – af(b)$$

$$X_1 = \frac{af(b)-bf(a)}{f(a)-f(b)}$$

Now $f(x_1)$ and f(a) are of opposite signs. If $f(x_1)\,f(a) < 0$ then x_2 lies between x_1 and a

$$X_2 = \frac{af(x1)-x1f(a)}{f(x1)-f(a)}$$

In the same way we get x_3, x_4,..........

This sequence will converge to the required root.

Problem 1:

Determine the root of $xe^x – 3 = 0$ correct to three decimal places using the method of false position.

Solution:

$$f(x) = x\,e^x – 3$$

$$f(0) = -3 = -ve$$

$$f(1)= 1e^1 – 3 = - 0.28172 = -ve$$

$$f(1.1) = (1.1)\,e^{1.1} – 3 = 0.30458 = +ve$$

Root lies between 1 and 1.1.

$$x^{(1)} = x_1 - \frac{f(x1)}{f(x2)-f(x1)}\,(x_2 – x_1)$$

$$x^{(1)} = 1 + \frac{0.28172}{(0.30458+0.28172)} \times 0.1$$

$$=1 + \frac{0.28172}{0.5863} \times 0.1$$

$$x^{(1)} = 1.04805$$

$f(x^1) = (1.04805)e^{1.04805} - 3 = -0.01087 = -ve$

Root lies between 1.04805 and 1.1

$x_1 = 1.04805 \qquad x_2 = 1.1$

$f(x_1) = -0.01087 \qquad f(x_2) = 0.30458$

$x^{(2)} = 1.04805 + \frac{0.01087}{(0.30458+0.01087)} \times 0.05195$

$x^{(2)} = 1.04984$

$f(x^2) = (1.04984)e^{1.04984} - 3 = -0.004 = -ve$

Root lies between 1.04984 and 1.1.

$x_1 = 1.04984 \qquad x_2 = 1.1$

$f(x_1) = -0.004 \qquad f(x_2) = 0.30458$

$x^{(3)} = 1.04984 + \frac{0.004}{(0.30458+0.004)} \times 0.05016$

$x^{(3)} = 1.05049$

$f(x^3) = (1.05049)e^{1.05049} - 3 = 0.003405 = +ve$

Root lies between 1.05049 and 1.04984.

$x_1 = 1.04984 \qquad x_2 = 1.05049$

$f(x_1) = -0.004 \qquad f(x_2) = 0.003405$

$x^{(4)} = 1.04984 + \frac{0.004}{(0.003405+0.004)} \times 6.5 \times 10^{-4}$

$x^{(4)} = 1.05019$

Better approximation to the root is 1.050.

Problem 2:

Compute the real root of $x \log_{10} x - 1.2 = 0$. Correct to five decimal places.

Solution:

$f(x) = x \log_{10} x - 1.2$

$f(0) = -1.2 = -ve$

$f(1) = -1.2 = -ve$

$f(2) = -0.59794 = -ve$

$f(3) = 0.23136 = +ve$

Root lies between 2 and 3.

$x_1 = 1$ $\quad\quad x_2 = 3$

$f(x_1) = -0.59794 \quad f(x_2) = 0.23136$

$x^{(1)} = 2 + \frac{0.59794}{(0.23136+0.59794)} \times 1$

$x^{(1)} = 2.72102$

$f(x^1) = 2.72102 \times \log_{10}(2.72102) - 1.2$

$= -0.01709 = -ve$

Root lies between 2.72102 and 3.

$x_1 = 2.72102$ $\quad\quad x_2 = 3$

$x^{(2)} = 2.72102 + \frac{0.01709}{(0.23136+0.01709)} \times 0.27898$

$x^{(2)} = 2.74021$

$f(x^2) = 2.74021 \times \log_{10}(2.74021) - 1.2$

$= -3.8 \times 10^{-4} = -0.00038 = -ve$

Root lies between 2.74021 and 3.

$x_1 = 2.74021$ $\quad\quad x_2 = 3$

$f(x_1) = -0.00038 \quad f(x_2) = 0.23136$

$x^{(3)} = 2.74021 + \frac{0.00038}{(0.23136+0.00038)} \times 0.25979$

$x^{(3)} = 2.74064$

$f(x^3) = 2.74064 \times \log_{10}(2.74064) - 1.2$

$= -5.3 \times 10^{-6} = -ve$

Root lies between 2.74064 and 3.

$x_1 = 2.74064$ $\quad\quad x_2 = 3$

$x^{(4)} = 2.74064 + \frac{0.0000053}{(0.23136+0.0000053)} \times 0.25936$

$x^{(4)} = 2.74065$

Value of the root is 2.74065 correct to five decimal places.

Problem 3:

Find the real root of the equation $x^3 - 9x + 1 = 0$, correct to 3 decimal places.

Solution: 2.943

Problem 4:

Find the real root of the equation $x^3 - x^2 - 2 = 0$ correct to 2 decimal places.

Solution : 1.69

Problem 5:

Find the root of $xe^x - 2 = 0$ which lies between 0 & 1 to four decimal places by false position method.

Solution : 0.8526

1.4. Newton Raphson Method:

Let y=f(x) be the simple equation and the root of f(x) =0 can be computed rapidly by a process called the Newton Raphson method.

Let x = x_0 be an approximate value of the root.

Let x = x_1 be the exact root f (x_1) = 0 ------------------1

Then $x_1 - x_0$ is small and equal to h.

$x_1 - x_0 = h$

$x_1 = x_0 + h$ --2

Substituting equation 2 in 1, we get

$f(x_0+h) = 0$

By Taylor's theorem.

$$f(x_0+h) = f(x_0) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots = 0$$

$$f(x_0) + \frac{h}{1!} f'(x_0) = 0$$

$$h = \frac{-f(x_0)}{f'(x_0)}$$

Therefore

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

And so on.

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Where n = 0, 1, 2……

Newton's method is applicable to the solution of equations involving algebraic functions as well as transcendental functions.

Problem 1:

Find by Newton's method, the root of the equation $e^x = 4x$, correct to three decimal places.

$f(x) = e^x - 4x$

$f(0) = 1 - 0 = 1$

$f(1) = e^1 - 4 = -1.2817$

$f(2) = e^2 - 8 = -0.6109$

$f(3) = e^3 - 12 = 8.085 = +ve$

Let us take $x_0 = 2.1$ as first approximate value

$f(x) = e^x - 4x$, $f(2.1) = -0.2338$

$f^1(x) = e^x - 4$, $f(2.1) = 4.1662$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Problem:1

Find by Newtons method, the root of the equation $\mathbf{e^x = 4x}$, correct to three decimal places.

Solution:

$f(x) = e^x - 4x$

$f(0) = 1-0 = 1$

$f(1) = e^1 - 4 = -1.2817$

$f(2) = e^2 - 8 = -0.6109$

$f(3) = e^3 - 12 = 8.085 = +ve$

Let us take $x_0 = 2.1$ as first approximation value

$f(x) = e^x - 4x \quad f(2.1) = -0.2338,$

$f^1(x) = e^x - 4 \quad f^1(2.1) = 4.1662$

$x_1 = x_0 - \frac{f(x0)}{f\prime(x0)}$

$x_1 = 2.1 + \frac{0.2338}{4.1662}$

$x_1 = 2.1561$

$x_2 = x_2 = x_1 - \frac{f(x1)}{f\prime(x1)}$

$f(x_1) = 0.01299$

$f(x_1) = 4.6374$

$x_2 = 2.1561 - \frac{0.01299}{4.6374}$

$x_2 = 2.1561 - (2.8011 \times 10^{-3})$

$f(x_2) = 3.5226 \times 10^{-5}$

$f^1(x_2) = 4.6132$

$x_3 = x_2 - \frac{f(x2)}{f\prime(x2)}$

$= 2.1533 - \frac{3.5226 \times 10-5}{4.6132}$

$x_3 = 2.1533 - (7.6359 \times 10^{-6})$

$x_3 = 2.1533$

From the values of x_2 and x_3 the values of the root are 2.153 correct to 3 decimal places.

Problem :2

Find by Newton Raphson method, the real root of **3x-cosx-1 = 0** to four decimal places.

Solution: 0.6666

Problem :3

Find by Newtons method the root of the equation $x^3-3x+1=0$ which is between 1 and 2 to 3 decimal places.

Solution: 1.532

Problem: 4

Find by Newtons method root of the equation $x^3-6x+4=0$ correct to four decimal places.

Solution: 0.7320

Problem: 5

Find by Newtons method, the root of the equation 2x-3sinx-5=0 to six decimal places.

1.5. Secant Method:

The secant method is a root-finding procedure in numerical analysis that uses a series of roots of secant lines to better approximate a root of a function f. Let us learn more about the second method, its formula, advantages and limitations, and secant method solved example with detailed explanations in this article.

What is a Secant Method?

The tangent line to the curve of $y = f(x)$ with the point of tangency $(x_0, f(x_0)$ was used in Newton's approach. The graph of the tangent line about $x = \alpha$ is essentially the same as the graph of $y = f(x)$ when $x_0 \approx \alpha$. The root of the tangent line was used to approximate α.

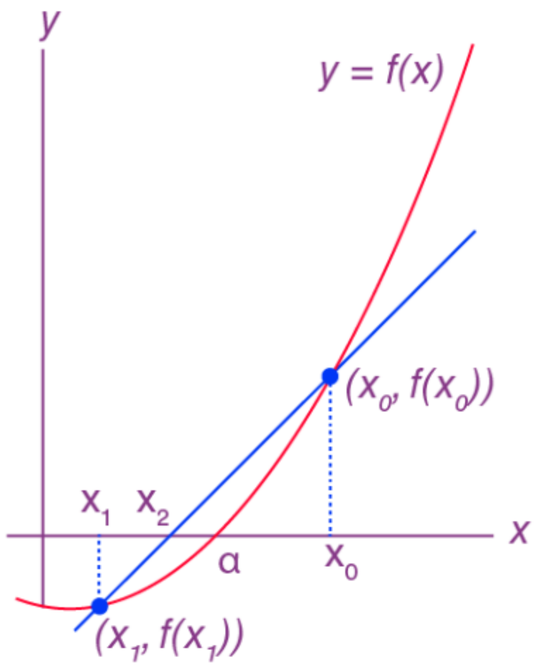
Consider employing an approximating line based on 'interpolation'. Let's pretend we have two root estimations of root α, say, x_0 and x_1. Then, we have a linear function

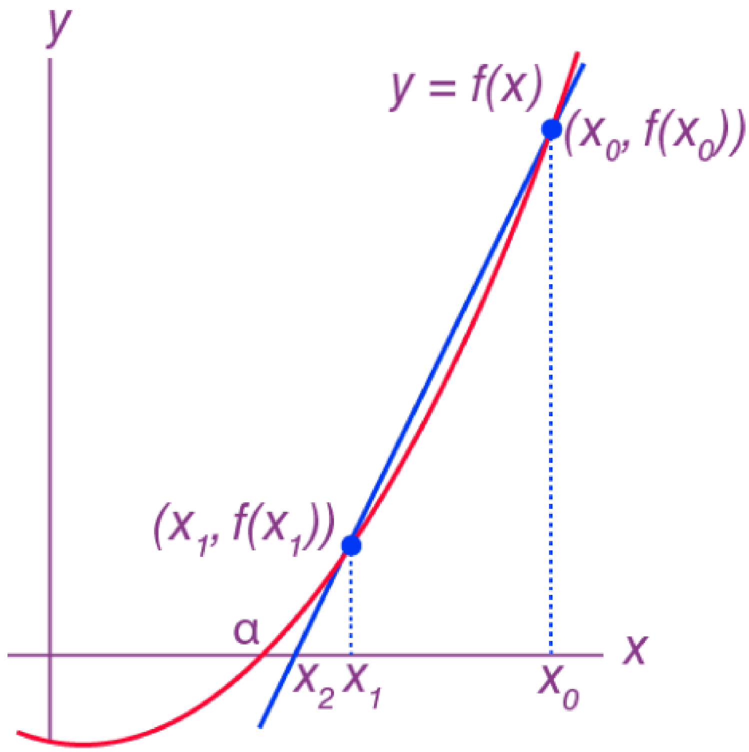
$$q(x) = a_0 + a_1x$$

using $$q(x_0) = f(x_0),\ q(x_1) = f(x_1).$$

This line is also known as a secant line. Its formula is as follows:

$$q(x) = \frac{(x_1 - x) f(x_0) + (x - x_0) f(x_1)}{x_1 - x_0}$$





The linear equation q(x) = 0 is now solved, with the root denoted by x_2. This results in

$$x_2 = x_1 - f(x_1) \cdot \frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

Let the above form be equation (1)

The procedure can now be repeated. Employ x_1 and x_2 to create a new secant line, and then use the root of that line to approximate α;…

Secant Method Steps

The secant method procedures are given below using equation (1).

Step 1: Initialization

x_0 and x_1 of α are taken as initial guesses.

Step 2: Iteration

In the case of n = 1, 2, 3, …,

$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

until a specific criterion for termination has been met (i.e., The desired accuracy of the answer or the maximum number of iterations has been attained).

Secant Method Convergence

If the initial values x_0 and x_1 are close enough to the root, the secant method iterates x_n and converges to a root of function f. The order of convergence is given by φ, where

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

Which is the golden ratio.

The convergence is particularly superlinear, but not really quadratic. This solution is only valid under certain technical requirements, such as f being two times continuously differentiable and the root being simple in the question (i.e., having multiplicity 1).

There is no certainty that the secant method will converge if the beginning values are not close enough to the root. For instance, if the function f is differentiable on the interval $[x_0, x_1]$, and there is a point on the interval where f' =0, the algorithm may not converge.

Secant Method Advantages and Disadvantages

The secant method has the following advantages:

- It converges quicker than a linear rate, making it more convergent than the bisection method.
- It does not necessitate the usage of the function's derivative, which is not available in a number of applications.

- Unlike Newton's technique, which requires two function evaluations in every iteration, it only requires one.

The secant method has the following drawbacks:

- The secant method may not converge.
- The computed iterates have no guaranteed error bounds.
- If $f_0(\alpha) = 0$, it is likely to be challenging. This means that when $x = \alpha$, the x-axis is tangent to the graph of $y = f(x)$.
- Newton's approach is more easily generalized to new ways for solving nonlinear simultaneous systems of equations.

1.6. Order of Convergence :

Let r be the root and x_n be the nth approximation to the root. Define the error as

$$\epsilon_n = r - x_n$$

If for large n we have the approximate relationship

$$|\epsilon n+1| = k|\epsilon n|p,$$

with k a positive constant, then we say the root-finding numerical method is of order p. Larger values of p correspond to faster convergence to the root. The order of convergence of bisection is one: the error is reduced by approximately a factor of 2 with each iteration so that

$$|\epsilon n+1| = 12|\epsilon n|.$$

1.7. Simultaneous Equation

Direct Methods:

1. Simultaneous linear algebraic equations occur in many fields.
2. Direct methods provide the exact solution of an equation system in a finite number of steps and try to solve the problem immediately.
3. When this method is used for finite arithmetic calculations usually obtain an approximation solution, generally due to rounding errors.

Gauss Elimination method:

This direct method is based on the elimination of the unknown one by one and transforming the given set of equations into a triangular form.

Now, we consider the general system of equations.

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$ --------- 1

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$ ----------2

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$ ----------3

We first form the augmented matrix containing the coefficient and the righthand side constants

$$\left(\begin{array}{ccc|c} a11 & a12 & a13 & b1 \\ a21 & a22 & a23 & b\ 2 \\ a31 & a32 & a33 & b3 \end{array}\right) \text{---------4}$$

First step:

Elimination of x_1 from equations 2 and 3.

To eliminate x_1 from second equation we multiply the first equation by $-\frac{a21}{a11}$ and add it to the equation 2. Similarly, to eliminate x_1 from third equation we multiply first equation by $-\frac{a31}{a11}$ and add it to third equation.

$$-\frac{a21}{a11} \quad -\frac{a31}{a11} \quad \left(\begin{array}{ccc|c} a11 & a12 & a13 & b1 \\ a21 & a22 & a23 & b2 \\ a31 & a32 & a33 & b3 \end{array}\right) \text{------------5}$$

Here $-\frac{a21}{a11}$ and $-\frac{a31}{a11}$ are called multipliers. It is clear that a_{11} not equal to 0. The first equation is called pivoted equation and the leading coefficient a_{11} is called first pivot.

Now equation 5 becomes

$$\left(\begin{array}{ccc|c} a11 & a12 & a13 & b1 \\ 0 & a22' & a23' & b2' \\ 0 & a32' & a33' & b3' \end{array}\right)$$

Second step:

Elimination of x_2 from the third equation

Now multiplying the second equation by $-\frac{a32'}{a22'}$ and adding it to third equation.

The table now becomes

$$-\frac{a32'}{a22'}\left(\begin{array}{ccc|c} a11 & a12 & a13 & b1 \\ 0 & a22' & a23' & b2' \\ 0 & a32' & a33' & b3' \end{array}\right)$$

$$\left(\begin{array}{ccc|c} a11 & a12 & a13 & b1 \\ 0 & a22' & a23' & b2' \\ 0 & 0 & a33' & b3' \end{array}\right)$$

From the reduced system of equations, the values of x_1, x_2, x_3 can be calculated.

Problem 1

Solve by gauss elimination method

2x+y+4z = 12

8x -3y+2z = 20

4x+11y-z = 33

$$\left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 9 & -9 & 9 \end{array}\right) \quad R_2 \rightarrow -4R_1+R_2$$

$R_3 \rightarrow -2R_1+R_3$

$$\left(\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 0 & -27 & -27 \end{array}\right) \quad R_3 \rightarrow \frac{9}{7} R_2 + R_3$$

-27z = -27

Z=1

-7y-14z = -28

-7y-14= -28

-7y = -28+14= -14

$Y=\frac{14}{7}$

Y = 2

2x+y+4z = 12

2x = 12-y-4z

= 12-2-4(1)

X= 3

Ans x = 3, y =2, z = 1

Problem :2

2x+y+z = 5

X+3y+2z = 4

-x+y+6z = 4

Solution: x = 2, y = 3, z = 3

Problem: 3

8x+2y-2z = 8

2x+y+9z = 12

X -8y+3z = -4

Solution: x = 1, y = 1, z = 1

Problem:4

2x+2y-z = 2

x-3y+z = -28

-x+ y = 14

Solution: x = -6, y = 8, z = 2

1.8. Gauss Jacobi Iteration Method:

Consider the system of equtions

$$a_1x + b_1y + c_1 = d_1$$

$$a_2x + b_2y + c_2 = d_2$$

$$a_3x + b_3y + c_3 = d_3$$

Suppose, in the above, in each equation, the coefficients of the diagonal terms are large, compared to other coefficients. This means that the equations are "ready" for iteration. Solving for x, y, z respectively.

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z)$$

$$y = \frac{1}{b_1}(d_2 - a_2x - c_2z)$$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y)$$

Suppose $x^{(0)}$, $y^{(0)}$, $z^{(0)}$ are initial estimates for the values of the unknowns x,y,z. Substituting these values in the right sides of (2), we have a system of first approximations or first iterates given by

$$x^{(1)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(0)} - c_1 z^{(0)}\right)$$

$$y^{(1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(0)} - c_2 z^{(0)}\right)$$

$$z^{(1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(0)} - b_3 z^{(0)}\right)$$

Substituting the values $x^{(1)}$, $y^{(1)}$, $z^{(1)}$ in the right sides of (2), we have the second approximations given by

$$x^{(2)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(1)} - c_1 z^{(1)}\right)$$

$$y^{(2)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(1)} - c_2 z^{(1)}\right)$$

$$z^{(1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(1)} - c_3 y^{(1)}\right)$$

If $x^{(r)}$, $y^{(r)}$, $z^{(r)}$ are the r^{th} iterates, then

$$x^{(r+1)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(r)} - c_1 z^{(r)}\right)$$

$$y^{(r+1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(r)} - c_2 z^{(r)}\right)$$

$$z^{(r+1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(r)} - c_3 y^{(r)}\right)$$

The process is continued till convergence is secured.

Note: It is clear that the procedure starts with an initial estimate for the values of x,y,z which are $x^{(0)}$, $y^{(0)}$, $z^{(0)}$. In the absence of any better estimate, they are taken as (0,0,0).

Example 1:

Solve, by Gauss – Jacobi method of iteration the equations

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

$$x + y + 54z = 110$$

Note: In the given equations, we find that the largest coefficient is attached to a different unknown. Also in each equation, the absolute value of the largest coefficient is greater than the sum of the remaining coefficients. So iteration method can be applied. From the equations, we have

$$x = \frac{1}{27}(85 - 6y - z)$$

$$y = \frac{1}{15}(72 - 2x - z)$$

$$z = \frac{1}{54}(10 - x - y)$$

We start the iteration by putting $x = 0 = y = z$ in the right sides of the equations, we have,

$$x^{(1)} = \frac{85}{27} = 3.14815$$

$$y^{(1)} = \frac{72}{15} = 48$$

$$z^{(1)} = \frac{110}{54} = 2.03704$$

Putting the values of $y^{(1)}$, $z^{(1)}$ in the right side of (1), we have , for second iteration,

$$x^{(2)} = 2.15693$$

$$y^{(2)} = 3.26913$$

$$z^{(2)} = 1.88985$$

Putting the values of $x^{(2)}$ $y^{(2)}$, $z^{(2)}$ in the right side of (1), we have , for third iteration,

$$x^{(3)} = 2.49167$$

$$y^{(3)} = 3.68525$$

$$z^{(3)} = 1.93655$$

Iteration	**x**	**y**	**z**
0	0	0	0
1	3.14815	4.8	2.03704
2	2.15693	3.26913	1.88985
3	2.49167	3.68525	1.93655
4	2.40093	3.54513	1.92265

5	2.43155	3.58327	1.92692
6	2.42323	3.57046	1.92565
7	2.42603	3.57395	1.92604
8	2.42527	3.57278	1.92593
9	2.42552	3.57310	1.92596
10	2.42546	3.57300	1.92595

We find the values in the 9^{th} and 10^{th} are close to each other. They are pratically the same, to 4 decimal place so we can stop the iteration process.

The values are **x = 2.4225, y = 3.5730, z = 1.9260**

Exercise:

1. Solve, by Gauss – Jacobi method of iteration the equations

$$4x + y = 3$$
$$-x + 3y = 7$$

Answer: x = 1.4545, y = 2.8182

2. Solve, by Gauss – Jacobi method of iteration the equations

$$10x + 2y - z = 27$$
$$-3x - 6y + 2z = -61.5$$
$$x + y + 5z = -21.5$$

Answer: x = 0.5000, y = 8.0000, z = -6.0000

1.9. Gauss Seidel Iterative method:

Indirect Method:

1. Indirect methods are those in which the solution is got by successive approximation.
2. Thus in an indirect or iterative method, the amount of computation depends on the degree of accuracy required.
3. This method of iteration is not applicable to all system of equations.
4. In order that iteration may succeed, each equation of the system must contain one large coefficient and the large coefficient must be attached to a different unknown in the equation. When the large coefficients are along the leading diagonal of the matrix this is possible. When the equations are in this form, they are soluble the indirect method.

Gauss Seidel Iterative method:

Consider the system of equations

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$ ------------- 1

$a_3 + b_3y + c_3z = d_3$

$$x = \frac{1}{a1}(d_1 - b_1y - c_1z)$$

$$y = \frac{1}{b2}(d_2 - a_2x - c_2z) \text{ -------------2}$$

$$z = \frac{1}{c3}(d_3 - a_3x - b_3y)$$

Substituting $y^{(0)}$, $z^{(0)}$ in the first equation

$$x^{(1)} = \frac{1}{a1}(d_1 - b_1 y^{(0)} - c_1z^{(0)})$$

then we substitute $x^{(1)}$ for x and $z^{(0)}$ for z in second equation $y^{(1)} = \frac{1}{b2}(d_2 - a_2x^{(1)} - c_2z^{(0)})$

then we substitute $x^{(1)}$ for x and $y^{(1)}$ for y we get

$$z^{(1)} = \frac{1}{c3}(d_3 - a_3x^{(1)} - b_3y^{(1)})$$

thus, as soon as a new value for a variable is found, it is used immediately in the following equations

if $x^{(r)}$, $y^{(r)}$, $z^{(r)}$ are the rth iterates, then

$$x^{(r+1)} = \frac{1}{a1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b2}(d_2 - a_2 x^{(r+1)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c3}(d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)})$$

*The progress is continued till convergence is secured

*Since the current values of the unknowns at each stage of iteration are used in proceeding to the next stage of iteration, the convergence will be more rapid.

Problem:1

Solve by gauss Seidel method of iteration.

$10x-2y+z = 12$

$x+9y-z = 10$

$2x-y+11z = 20$

$$x= \frac{1}{10}(12x+2y-z) \text{ ---------1}$$

$$y = \frac{1}{9}(10-x+z) \text{ ------------2}$$

$$z = \frac{1}{11}(20-2x+y) \text{ ----------3}$$

$$x^{(1)} = \frac{1}{10}(12+0) = 1.2$$

$y^{(1)} = \frac{1}{9}(10-1.2+0) = 0.977778$

$z^{(1)} = \frac{1}{11}(20-2(1.2) + 0.977778$

$= 1.688889$

$x^{(2)} = \frac{1}{10}(12+2(0.977778)-1.688889)$

$= 1.22667$

$y^{(2)} = \frac{1}{9}(10-1.22667+1.688889)$

$= 1.16247$

$z^{(2)} = \frac{1}{11}(20-2(1.22667) +1.16247)$

$z^{(2)} = \frac{1}{11}(18.70913) = 1.70083$

$x^{(3)} = \frac{1}{10}(12.624411)$

$x^{(3)} =1.262411$

$y^{(3)} = \frac{1}{9}(10-1.2624+1.70083)$

$y^{(3)} = 1.15982$

$z^{(3)} = \frac{1}{11}(20-2(1.262411)+(1.15982)$

$z^{(3)} = 1.69409$

$x^{(4)} = \frac{1}{10}(12+2(1.15982)-1.69409)$

$x^{(4)} = 1.26256$

$y^{(4)} = \frac{1}{9}(10-1.26256+1.69409)$

$y^{(4)} = 1.15906$

$z^{(4)} = \frac{1}{11}(20-2(1.26256) +1.15906)$

$z^{(4)} = 1.69399$

$x^{(5)} = \frac{1}{10}(12+2(1.15906)-1.69399)$

$x^{(5)} = 1.262413$

$y^{(5)} = \frac{1}{9}(10-1.262413+1.69399)$

$y^{(5)} = 1.15906$

$z^{(5)} = \frac{1}{11}(20-2(1.262413) +1.15906)$

$z^{(5)} = 1.69402$

$x^{(6)} = \frac{1}{10}(12+2(1.15906)-1.69402)$

$x^{(6)} =1.26241$

$y^{(6)} = \frac{1}{9}(10-1.26241+1.69402)$

$y^{(6)} = 1.15906$

$z^{(6)} = \frac{1}{11}(20-2(1.26241) +1.15906)$

$z^{(6)} = 1.69402$

Form these, Solution:

x = 1.26241

y = 1.15906

z = 1.69402

Problem 2

x+17y-2z = 48

30x-2y+3z = 75

2x+2y=18z = 30

Solution

x = 2.57958

y = 2.79758

z = 1.06920

Problem 3

8x-3y+2z = 20

4x+11y-z = 33

6x+3y+12z = 35

Solution

X=3.0167

Y=1.9858

Z=0.9118

Problem 4

28x+4y-z = 32

2x+17y+4z = 35

X+3y+10z = 24

Solution

X = 0.9935

Y= 1.5069

Z = 1.8485

1.10. Inverse of a matrix using Gauss elimination Method:

Consider a matrix

$$A= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Let A^{-1} be the inverse of the matrix then $AA^{-1} = I$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Where I is the identity matrix

Therefore Augmented Matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{---} 1$$

Solving this equation by Gauss elimination method.

$R_2 \longrightarrow R_2 - \left(\frac{a_{21}}{a_{11}}\right) R_1$

$R_3 \longrightarrow R_3 - \left(\frac{a_{31}}{a_{11}}\right) R_1$

Equation 1 becomes

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^1 & a_{23}^1 \\ 0 & a_{32}^1 & a_{33}^1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 \end{bmatrix} \text{-----------------------------------} 2$$

Where $a_{22}^1 = a_{22} - a_{12}\frac{a_{21}}{a_{11}}$

$a_{23}^1 = a_{23} - a_{13}\frac{a_{21}}{a_{11}}$

$a_{32}^1 = a_{32} - a_{12}\frac{a_{31}}{a_{11}}$

$a_{33}^1 = a_{33} - a_{12}\frac{a_{31}}{a_{11}}$

Now to eliminate a_{32}^1 from equa 2 and the multiplier is $-\frac{a_{32}^1}{a_{22}^1}$

Multiply equ 2 by - $\frac{a_{32}^1}{a_{22}^1}$ and adding to the row R2. We get augmented matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^1 & a_{23}^1 \\ 0 & a_{32}^1 & a_{33}^1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 \\ -\frac{a_{31}}{a_{11}} & +\frac{a_{21}}{a_{11}} x \frac{a_{32}^1}{a_{22}^1} 0 & 1 \end{bmatrix} \text{-------------------3}$$

Where

$$a''_{33} = a'_{33} - a'_{23} \text{ x } \frac{a_{32}^1}{a_{22}^1}$$

From equ 3, we get three augmented matrices and it will be in the following form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{'} & a_{23}^1 \\ 0 & 0 & a_{33}^{''} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{21}}{a_{11}} & 0 & 0 \\ -\frac{a_{31}}{a_{11}} & +\frac{a_{21}}{a_{11}} x \frac{a_{32}^1}{a_{22}^1} 0 & 0 \end{bmatrix} \text{-----------------------4}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{'} & a_{23}^1 \\ 0 & 0 & a_{33}^{''} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{--5}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{'} & a_{23}^1 \\ 0 & 0 & a_{33}^{''} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{--6}$$

By solving these matrices finally we obtain the inverse of the matrix.

Problem 1:

Find by Gauss elimination, the inverse of the matrix.

$$\begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{---}1$$

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & \frac{5}{2} & -2 \\ 0 & -\frac{9}{4} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{4} & 0 & 1 \end{bmatrix} \text{---}2$$

$R_2 \longrightarrow R_2 - \left(\frac{R_1}{2}\right)$

$R_3 \longrightarrow R_3 - \left(\frac{R_1}{4}\right)$

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & \frac{5}{2} & -2 \\ 0 & 0 & -\frac{3}{10} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{7}{10} & \frac{9}{10} & 1 \end{bmatrix} \text{--------------------------}3$$

$R_3 \longrightarrow R_3 + \left(\frac{9}{10}\right) R_2$

Equation 3 is equivalent to

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & \frac{5}{2} & -2 \\ 0 & 0 & -\frac{3}{10} \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{7}{10} \end{bmatrix} \text{-------------------------------}4$$

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & \frac{5}{2} & -2 \\ 0 & 0 & -\frac{3}{10} \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ \frac{9}{10} \end{bmatrix} \text{-------------------------------}5$$

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & \frac{5}{2} & -2 \\ 0 & 0 & -\frac{3}{10} \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{-------------------------------}6$$

Solving Equation 4, 5, 6 by back substitution method we get,

From 4,

$-\frac{3}{10}\ Z = -\ \frac{7}{10}$

$$Z = \frac{7}{3}$$

$$\frac{5}{2}y - 2z = -\frac{1}{2}$$

$$\frac{5}{2}y - 2\left(\frac{7}{3}\right) = -\frac{1}{2}$$

$$\frac{5}{2}y = -\frac{1}{2} + \frac{14}{3}$$

$$\frac{5}{2}y = \frac{25}{6}$$

$$y = \frac{5}{3}$$

$4x + y + 2z = 1$

$4x = 1 - y - 2z$

$= 1 - 5/3 - 14/3$

$= 1 - 19/3 = -16/3$

$$x = -4/3$$

The Solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4/3 \\ 5/3 \\ 7/3 \end{bmatrix}$$

From equ 5,

$$-\frac{3}{10}z = \frac{9}{10}$$

$$z = -3$$

Same procedure to find the values of y, x and from equation 6, z, y, x
----

Finally, from these inverse of the matrix is

$$\begin{bmatrix} -4/3 & 2 & 7/3 \\ 5/3 & -2 & -8/3 \\ 7/3 & -3 & -10/3 \end{bmatrix}$$

Problem 2:

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

Ans: $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$

Problem 3:

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

Ans: $\begin{bmatrix} -3 & 5/2 & -1/2 \\ 12 & -17/2 & 3/2 \\ -5 & 7/2 & -1/2 \end{bmatrix}$

Curve Fitting – Interpolation:

Curve fitting: Method of least squares – straight line, fitting a parabola, fitting y= axn, y = aebx type curves – Interpolation: Polynomial Interpolation – Lagrange polynomial – Newton polynomial – Forward and Backward differences – Gregory Newton forward and backward interpolation formula for equal intervals – Divided difference – properties of divided differences – Newton's divided differences formula – Lagrange's interpolation formula for unequal interval.

2.1. Curve fitting:

Several equations of different types can be obtained to express the given data approximately. But the problem is to find the equation of the curve of "best fit" which may be most suitable for predicting the unknown values. The process of finding such an equation of "best fit" known as curve-fitting.

2.2. Method of Least Square:

For clarity, suppose it is required to fit the curve $y = a+bx+cx^2$ to a given set of observations $(x_1, y_1), (x_2, y_2) \dots (x_5, y_5)$. For any x_1, the observed value is y_i and the expected value is $\eta_i = a+ bx_i+cx^2_i$ so that the error $e_i = y_i - \eta_i$

$\therefore$ Thesum of the square of these errorsis

$E = e_1^2+e_2^2+ \dots\dots e_5^2$

$= [y_1-(a+bx_1 + cx_1^2)]^2 +[y_2-bx_2+cx_2^2)]^2 +\dots+ [y_5-(a+bx_5 +cx_5^2)]^2$

For E to be minimum, we have

$$\frac{\partial E}{\partial a} = 0 = -2[y_1 - (a + bx_1 + cx_1^2)]^2 - 2[y_2 - (a + bx_2 + cx_2^2)]-\dots\dots- 2[y_5 - (a + bx_5 + cx_5^2)]^2 \quad (1)$$

$$\frac{\partial E}{\partial b} = 0 = -2x_1[y_1 - (a + bx_1 +cx_1^2)] - 2x_2 [y_2 - (a + bx_2 + cx_2^2)] -\dots- 2[y_5 - (a + bx_5 + cx_5^2)] \quad (2)$$

$$\frac{\partial E}{\partial b} = 0 = -2x_1^2[y_1 - (a + bx_1 + cx_1^2)] - 2x_2^2[y_2 - (a + bx_2 + cx_2^2)] -\dots -2x_5^2[y_5 - (a +bx_5 + cx_5^2)](3)$$

Equation (1) simplifies to

$y_1 + y_2 + \dots + y_5 = 5a + b\,(x_1 + x_2 + \dots + x_5) + c\,(x_1^2 + x_2^2 + \dots + x_5^2)$

i.e, $$\sum yi = 5a + b\,\sum xi + cxi^2 \quad (4)$$

Equation (2) becomes

$x_1y_1 + x_2y_2 + \dots + x_5y_5 = a\,(x_1 + x_2 + \dots + x_5) + b\,(x_1^2 + x_2^2 + \dots + x_5^2) + c\,(x_1^3 + x_2^3 + \dots + x_5^3)$

i.e., $\sum x_i y_i = a\sum x_I + b\sum x_i^2 + x\sum x_i^3$ (5)

Similarly (3) simplifies to

$$\sum x_i^2 y_i = a\sum x_i^2 + b\sum x_i^3 + c\sum x_i^4 \quad (6)$$

The equations (4), (5) and (6) are known as normal equations and can be solved as simultaneous equations in a,b,c . The values of these constants when substituted in (1) give the desired curve of best it.

Working procedures:

(a) To fit the straight-line $y = a+bx$

(i)Substitute the observed set of n values in the equation.

(ii)Form normal equations for each constant, i.e., $\sum y = na + b\sum x$, $\sum xy = a\sum x + b\sum x^3$

[The normal equation for the unknown a is obtained by multiplying the equation by the coefficient a and adding. The normal equation by the coefficient of a and adding. The normal equation for b is obtained by multiplying the equations by the coefficient of b (i.e.) and adding.]

(iii) Solve these normal equations as simultaneous equation for a and b.

(iv) Substitute the values of and b in $y = a+bx$, which is the required line of best fit.

(b) To fit the parabola $y = a+bx+cx^2$

(i) Form the normal equations $\sum y = na + b\sum x + c\sum x^2$

$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$ and $\sum x^2 y = a\sum x^2 + b\sum x^2 + c\sum x^4$

The normal equation for c has been obtained by multiplying the equations by the coefficient of c (i.e., x^2) and adding.

(ii) Solve these as simultaneous equations for a,b,c.

(iii) Substitute the values of a,b,c in $y = a+bx+cx^2$, which is the required parabola of best fit.

(c) In general, the curve $y = a+bx+cx^2+\ldots\ldots\ldots\ldots+kx^{m-1}$ can be fitted to a given data by erecting m normal equations.

EXAMPLE: 1

To find the linear law of the form $P = mW + c$, we will use the given data to calculate the slope m and intercept c of the line. Then, we can use the equation to compute P for $W = 150$ kg.

Steps:

Given

$P = [12,15,21,25]$, $W = [50,70,100,120]$

$$P = mW + c$$

We calculate m (the slope) using the formula:

$$m = \frac{\sum\left(W_i - \bar{W}\right)\left(P_i - \bar{P}\right)}{\sum\left(W_i - \bar{W}\right)^2}$$

where $\bar{W}$ and $\bar{P}$ are the mean values of W and P, respectively.

Compute c:

Once m is known, we can find c using:

$$c = \bar{P} - m\bar{W}$$

Calculate P for $W = 150$:

Substitute $W = 150$ into $P = mW + c$.

Let's compute this step by step.

The linear law connecting P and W is:

$$P = 0.188W + 2.276$$

For $W = 150$ kg, the corresponding P is:

$$P = 30.47 \text{ kg-wt (rounded to 2 decimal places).}$$

EXAMPLE: 2

To fit a straight line of the form $y = mx + c$ to the given data, we can use the **least-squares method**. This involves calculating the slope m and the intercept c. The formulas are:

$$m = \frac{\sum\left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)}{\sum\left(x_i - \bar{x}\right)^2}$$

$$c = \bar{y} - m\bar{x}$$

Where:

- $\bar{x}$ and $\bar{y}$ are the means of the x and y values, respectively.

Given data:

- $x = [6,7,7,8,8,8,9,9,10]$
- $y = [5,5,4,5,4,3,4,3,3]$

Let's compute the slope m, intercept c, and the equation of the line.

The equation of the straight line that best fits the given data is:

$$y = -0.5x + 8.0$$

Solution:

The problem involves deriving the equation of a straight line $y = ax + b$ that fits the given data using the **method of normal equations**.

Summary of the Solution:

Normal equations:

- $\sum y = a\sum x + 9b$
- $\sum xy = a\sum x^2 + b\sum x$

Given values:

$$\sum x = 72, \quad \sum y = 36, \quad \sum xy = 282, \quad \sum x^2 = 588$$

Substituting into the normal equations:

$$36 = 72a + 9b \quad \text{(divide through by 9 to simplify to)} \quad 8a + b = 4 \quad \text{(equation 1)}$$

$$282 = 588a + 72b \quad \text{(divide through by 6 to simplify to)} \quad 98a + 12b = 47 \quad \text{(equation 2)}$$

Solving equations (1) and (2):

- Multiply equation (1) by 12: $96a + 12b = 48$
- Subtract this from equation (2): $98a + 12b - 96a - 12b = 47 - 48$, yielding $2a = -1$, so $a = -0.5$.

Substituting $a = -0.5$ into equation (1):

$$8(-0.5) + b = 4 \quad \Rightarrow \quad -4 + b = 4 \quad \Rightarrow \quad b = 8$$

$$y = -0.5x + 8$$

This matches the result derived earlier, confirming that the line of best fit is indeed $y = -0.5x + 8$.

EXAMPLE: *3*

To fit a second-degree parabola of the form $y = a + bx + cx^2$ to the given data:

Data:

$$x = [0,1,2,3,4], \quad y = [1,1.8,1.3,2,6.3]$$

Transformation:

We set:

$$u = x - 2 \quad \text{and} \quad v = y$$

This transforms the parabola $y = a + bx + cx^2$ into:

$$v = A + Bu + Cu^2$$

Normal Equations:

Using the least-squares method, the normal equations for this transformed parabola are:

$$\sum v = A\sum 1 + B\sum u + C\sum u^2$$

$$\sum uv = A\sum u + B\sum u^2 + C\sum u^3$$

$$\sum u^2 v = A\sum u^2 + B\sum u^3 + C\sum u^4$$

Compute $u = x - 2$, $v = y$, and the required sums: $\sum u$, $\sum u^2$, $\sum u^3$, $\sum u^4$, $\sum v$, $\sum uv$, and $\sum u^2 v$.

Solve the resulting simultaneous equations for A, B, and C.

Substitute$A$,B, and C back into the transformed parabola $v = A + Bu + Cu^2$.

Rewrite the equation in terms of x:

$y = A + B(x - 2) + C(x - 2)^2$.

Let's compute this step by step.

The coefficients of the second-degree parabola are:

$$A = 1.31, \quad B = 1.08, \quad C = 0.59 \text{ (rounded to 2 decimal places)}$$

The equation of the parabola in terms of x is:

$$y = 0.59x^2 - 1.26x + 1.49$$

EXAMPLE 4

Fit a second-degree parabola to the following data:

x = 1.0 1.5 2.0 2.5 3.0 3.5 4.0

y = 1.1 1.3 1.6 2.0 2.7 3.4 4.1

Explanation of the Solution:

- To simplify computations, the origin is shifted to $(2.5,0)$, and the unit is changed to 0.5. This transform x to X by the relation $X = 2x - 5$.
- The parabola $y = a + bx + cx^2$ becomes $y = A + BX + CX^2$ after the transformation
- The values of X, Xy, X^2, X^2y, X^3, and X^4 are computed using the given x and y.
- The totals are:

$$\sum X = 0, \quad \sum X^2 = 28, \quad \sum X^3 = 0, \quad \sum X^4 = 196,$$

$$\sum y = 16.2, \quad \sum Xy = 14.3, \quad \sum X^2 y = 69.9.$$

Using the least-squares method, the equations are:

$$7A + 28C = 16.2$$

$$28A + 196C = 69.9$$

- From $7A + 28C = 16.2$, substitute $A = 2.07 - 4C$ into $28A + 196C = 69.9$ and solve for C, giving $C = 0.061$.
- Substitute $C = 0.061$ back to find $A = 2.07$.

The coefficients are:

$$A = 2.07, \quad B = 0.511, \quad C = 0.061$$

- In terms of X:

$$y = 2.07 + 0.511X + 0.061X^2$$

- Replacing $X = 2x - 5$, the equation becomes:

$$y = 2.07 + 0.511(2x - 5) + 0.061(2x - 5)^2$$

- Simplifying:

$$y = 1.04 - 0.198x + 0.244x^2$$

The required parabola of best fit is:

$$y = 1.04 - 0.198x + 0.244x^2$$

EXAMPLE: 5

Fit a second-degree parabola to the following data:

x: 1989 1990 1991 1992 1993 1994 1995 1996 1997

y: 352 356 357 358 360 361 361 360 35

Solution

The given problem fits a second-degree parabola of the form $y = a + bx + cx^2$ to the data using transformations $u = x - 1993$ and $v = y - 357$. The steps to derive the equation are as follows:

The transformations are:

$$u = x - 1993, \quad v = y - 357$$

This simplifies the computations by shifting the origin to $(1993,357)$.

Using the given data:

$$\sum u = 0, \quad \sum u^2 = 60, \quad \sum u^3 = 0, \quad \sum u^4 = 708,$$
$$\sum v = 11, \quad \sum uv = 51, \quad \sum u^2 v = -9.$$

The parabola $v = A + Bu + Cu^2$ leads to the following normal equations

$$\sum u = 0, \quad \sum u^2 = 60, \quad \sum u^3 = 0, \quad \sum u^4 = 708,$$
$$\sum v = 11, \quad \sum uv = 51, \quad \sum u^2 v = -9.$$

- From the first equation:

$$A = \frac{11 - 60C}{9}$$

- Substitute A into the third equation:

$$60\left(\frac{11 - 60C}{9}\right) + 708C = -9$$

Simplify:

$$660 - 400C + 708C = -81$$

- Substitute C into $A = \frac{11-60C}{9}$:

$$A = \frac{11 - 60(-2.405)}{9} = \frac{11 + 144.3}{9} = 17.247$$

Thus:

$$A = 17.247, \quad B = 0.85, \quad C = -2.405$$

Transforming back to x using $u = x - 1993$:

$$v = 17.247 + 0.85u - 2.405u^2$$

$$y - 357 = 17.247 + 0.85(x - 1993) - 2.405(x - 1993)^2$$

Expand and simplify:

$$y = 0.85x - 1694.05 - 2.405x^2 + 1065.52x - 1061792.32$$

Combine terms:

$$y = -2.405x^2 + 1066.37x - 1065186.37$$

The required parabola of best fit is:

$$y = -2.405x^2 + 1066.37x - 1065186.37$$

This section describes methods for fitting nonlinear curves by transforming them into linear equations using logarithmic transformations. Below is a summary of the fitting approaches for these three types of curves:

1. Curve: $y = ax^b$

- **Transformation:** Take the logarithm of both sides:

$$\log_{10}y = \log_{10}a + b\log_{10}x$$

Let:

$$X = \log_{10} x, \quad Y = \log_{10} y, \quad A = \log_{10} a$$

The equation becomes:

$$Y = A + bX$$

- **Normal equations:**

$$\sum Y = nA + b\sum X, \quad \sum XY = A\sum X + b\sum X^2$$

- Solve for A and b, and calculate a using $A = \log_{10} a$, so $a = 10^A$.

2. **Curve:** $y = ae^{bx}$ (Exponential Curve)

- **Transformation:** Take the logarithm of both sides:

$$\log_{10} y = \log_{10} a + bx\log_{10} e$$

Let:

$$Y = \log_{10} y, \quad A = \log_{10} a, \quad B = b\log_{10} e$$

The equation becomes:

$$Y = A + Bx$$

- **Normal equations:**

$$\sum Y = nA + B\sum x, \quad \sum xY = A\sum x + B\sum x^2$$

- Solve for A and B, and calculate a and b:

$$a = 10^A, \quad b = \frac{B}{\log_{10} e}$$

3. **Curve:** $x^a y = b$ (Gas Equation, e.g., $pv^\gamma = k$)

- **Transformation:** Take the logarithm of both sides:

$$\log_{10} x + a\log_{10} y = \log_{10} b$$

Rearrange:

$$\frac{\log_{10}x}{a} + \log_{10}y = \frac{\log_{10}b}{a}$$

Let:

$$X = \log_{10}x, \quad Y = \log_{10}y, \quad A = \frac{\log_{10}b}{a}, \quad B = -\frac{1}{a}$$

The equation becomes:

$$Y = A + BX$$

- **Normal equations:**

$$\sum Y = nA + B\sum X, \quad \sum XY = A\sum X + B\sum X^2$$

- Solve for A and B, and calculate a and b:

$$a = -\frac{1}{B}, \quad b = 10^{aA}$$

This systematic approach simplifies fitting complex curves to real-world data. Let me know if you'd like help solving specific examples or visualizing these fits!

EXAMPLE: 6

An experiment gave the following values:

v (ft/min): 350 400 50 600

t (min): 61 26 7 2.6

To solve for the best values of a and b in the equation $v = at^b$, we can use the method of logarithmic transformations. Here's a step-by-step solution:

Given:

We know the relationship between v (velocity) and t (time) is $v = at^b$. We are asked to find the values of a and b.

The given data is:

- v (ft/min): 350, 400, 500, 600

- t (min): 61, 26, 7, 2.6

Take the logarithm of both sides of the equation $v = at^b$ to linearize the equation:

$$\log_{10} v = \log_{10} a + b\log_{10} t$$

Let:

- $X = \log_{10} t$
- $Y = \log_{10} v$
- $A = \log_{10} a$

The equation becomes:

$$Y = A + bX$$

This is now a linear equation in X and Y. We can solve for A and b using the normal equations.

From the transformed equation, the normal equations are:

$$\sum Y = 4A + b\sum X \quad \text{(i)}$$

$$\sum XY = A\sum X + b\sum X^2 \quad \text{(ii)}$$

Using the given data, calculate the following values:

v	t	$X = \log_{10} t$	$Y = \log_{10} v$	XY	X^2
350	61	1.7853	2.5441	4.542	3.187
400	26	1.4150	2.6021	3.682	2.002
500	7	0.8451	2.6990	2.281	0.714
600	2.6	0.4150	2.7782	1.153	0.172
Total		**4.4604**	**10.6234**	**11.658**	**6.075**

Now substitute the summation values into the normal equations.

1. From equation (i):

$$4A + b \cdot 4.46 = 10.623$$

2. From equation (ii):

$$4.46A + b \cdot 6.075 = 11.658$$

These give us the system of equations:

$$4A + 4.46b = 10.623 \quad (1)$$

$$4.46A + 6.075b = 11.658 \quad (2)$$

Solve the system of equations (1) and (2) to find A and b.

- Multiply equation (1) by 4.46 to eliminate A:

$$(4)(4.46)A + (4.46)(4.46)b = 10.623(4.46)$$

$$17.84A + 19.91b = 47.43 \quad (3)$$

- Subtract equation (2) from equation (3) to solve for b:

$$17.84A + 19.91b - (4.46A + 6.075b) = 47.43 - 11.658$$

$$13.38A + 13.835b = 35.772$$

Solve this to find:

$$A = 2.845, \quad b = -0.1697$$

We know that:

$$A = \log_{10} a$$

So:

$$a = 10^A = 10^{2.845} = 699.8$$

Thus, the best possible values of a and b are:

$$a \approx 699.8, \quad b \approx -0.1697$$

$$v = 699.8t^{-0.1697}$$

EXAMPLE: 7

Predict the mean radiation dose at an altitude of 3000 feet by fitting an exponential curve to the given data:

Altitude (x): 50 450 780 1200 4400 4800 5300

Dose of radiation (y): 28 30 32 36 51 58 69

To predict the mean radiation dose at an altitude of 3000 feet by fitting an exponential curve to the given data, we can proceed with the following steps:

Given Data:

- Altitude x: 50, 450, 780, 1200, 4400, 4800, 5300
- Dose of radiation y: 28, 30, 32, 36, 51, 58, 69

We are given that the relationship between radiation dose y and altitude x follows an exponential curve:

$$y = ab^x$$

Taking the logarithm of both sides, we get:

$$\log_{10}y = \log_{10}a + x\log_{10}b$$

Let:

- $Y = \log_{10}y$
- $A = \log_{10}a$
- $B = \log_{10}b$

This transforms the equation into a linear form:

$$Y = A + Bx$$

The normal equations for the linearized form are:

$$\sum Y = 7A + B\sum x \quad \text{(i)}$$

$$\sum xY = A\sum x + B\sum x^2 \quad \text{(ii)}$$

x	y	$Y = \log_{10} y$	xY	x^2
50	28	1.447158	72.3579	2500
450	30	1.477121	664.7044	202500
780	32	1.505150	1174.0170	608400
1200	36	1.556303	1867.5636	1440000
4400	51	1.707570	7513.3080	19360000
4800	58	1.763428	8464.4544	23040000
5300	69	1.838849	9745.8997	28090000
Total		**11.295579**	**29502.305**	**72743400**

Substitute the summation values into the normal equations:

From equation (i):

$$11.295579 = 7A + 16980B$$

From equation (ii):

$$29502.305 = 16980A + 72743400B$$

We now solve the system of two linear equations:

$$11.295579 = 7A + 16980B$$

$$29502.305 = 16980A + 72743400B$$

Using standard methods (e.g., substitution or matrix methods), we solve for A and B:

- $A = 1.4521015$
- $B = 0.0000666289$

Now, we can use the values of A and B to predict the radiation dose at an altitude of 3000 feet.

From the equation $Y = A + Bx$, we have:

$$Y = 1.4521015 + 0.0000666289 \times 3000$$

$$Y = 1.4521015 + 0.1998867 = 1.6519882$$

Now, take the antilog to find y:

$$y = 10^{1.6519882} \approx 44.874$$

Thus, the predicted mean radiation dose at an altitude of 3000 feet is approximately **44.9** units.

EXAMPLE: 8

Fit a curve of the form $y = ae^{bx}$ to the following data:

x: 0 1 2 3

y: 1.05 2.10 3.85 8.30

To fit a curve of the form $y = ae^{bx}$ to the given data, we can proceed with the following steps:

Given Data:

- x: 0, 1, 2, 3
- y: 1.05, 2.10, 3.85, 8.30

We are given the equation $y = ae^{bx}$. Taking the logarithm of both sides:

$$\log_{10} y = \log_{10} a + bx \log_{10} e$$

Since $\log_{10} e$ is a constant, we rewrite the equation as:

$$Y = A + Bx$$

Where:

- $Y = \log_{10} y$
- $A = \log_{10} a$

- $B = b\log_{10}e$

This transforms the equation into a linear form.

The normal equations for the linearized form are:

$$\sum Y = 4A + B\sum x \quad \text{(i)}$$

$$\sum xY = A\sum x + B\sum x^2 \quad \text{(ii)}$$

x	y	$Y = \log_{10}y$	x^2	xY
0	1.05	0.0212	0	0
1	2.10	0.3222	1	0.3222
2	3.85	0.5855	4	1.1710
3	8.30	0.9191	9	2.7573
Total		**1.8480**	**14**	**4.2505**

Now, we can substitute these summations into the normal equations.

From equation (i):

$$\sum Y = 4A + B\sum x \quad \Rightarrow \quad 1.8480 = 4A + 6B$$

From equation (ii):

$$\sum xY = A\sum x + B\sum x^2 \quad \Rightarrow \quad 4.2505 = 6A + 14B$$

So we have the system of linear equations:

2. $4A + 6B = 1.8480$
3. $6A + 14B = 4.2505$

We can solve this system of equations using substitution or matrix methods. After solving, we get:

- $A = 0.0185$
- $B = 0.2956$

From the equation $A = \log_{10} a$, we can calculate a:

$$a = 10^A = 10^{0.0185} = 1.0186$$

From $B = b\log_{10} e$, we calculate b as:

$$b = \frac{B}{\log_{10} e} = \frac{0.2956}{\log_{10} e} = 0.6806$$

Thus, the curve of best fit is:

$$y = 1.0186e^{0.6806x}$$

This is the required exponential curve that best fits the given data.

EXAMPLE: 9

The pressure and volume of a gas are related by the equation pVγ = k, γ and k being constants. Fit this equation to the following set of observations:

p (kg/cm2): 0.5 1.0 1.5 2.0 2.5 3.0

V (litres): 1.62 1.00 0.75 0.62 0.52 0.46

To fit the equation $pV^\gamma = k$ to the given data, we can follow these steps:

Given Data:

- Pressure p (kg/cm^2): 0.5, 1.0, 1.5, 2.0, 2.5, 3.0
- Volume V (litres): 1.62, 1.00, 0.75, 0.62, 0.52, 0.46

The equation $pV^\gamma = k$ can be transformed by taking logarithms:

$$\log_{10} p + \gamma \log_{10} V = \log_{10} k$$

Let:

- $X = \log_{10} V$
- $Y = \log_{10} p$
- $A = \log_{10} k$

Thus, the equation becomes:

$$Y = A + BX$$

Where $B = \gamma$.

The normal equations for the linearized form are:

$$\sum Y = 6A + B\sum X \quad \text{(i)}$$

$$\sum XY = A\sum X + B\sum X^2 \quad \text{(ii)}$$

p	V	$X = \log_{10} p$	$Y = \log_{10} V$	XY	X^2
0.5	1.62	-0.3010	0.2095	-0.0630	0.0906
1.0	1.00	0.0000	0.0000	-0.0000	0.0000
1.5	0.75	0.1761	-0.1249	-0.0220	0.0310
2.0	0.62	0.3010	-0.2076	-0.0625	0.0906
2.5	0.52	0.3979	-0.2840	-0.1130	0.1583
3.0	0.46	0.4771	-0.3372	-0.1609	0.2276
Total		**1.0511**	**-0.7442**	**-0.4214**	**0.5981**

From equation (i):

$$\sum Y = 6A + B\sum X \quad \Rightarrow \quad -0.7442 = 6A + 1.0511B$$

From equation (ii):

$$\sum XY = A\sum X + B\sum X^2 \quad \Rightarrow \quad -0.4214 = 1.0511A + 0.5981B$$

Thus, we have the system of equations:

$$6A + 1.0511B = -0.7442$$

$$1.0511A + 0.5981B = -0.4214$$

Solving the system of equations, we find:

- $A = 0.0132$
- $B = -0.7836$

From $A = \log_{10} k$, we can calculate k as:

$$k = 10^A = 10^{0.0132} = 1.039$$

Also, from $B = \gamma$, we can calculate γ as:

$$\gamma = -\frac{1}{B} = \frac{1}{0.7836} = 1.276$$

Thus, the equation of best fit is:

$$pV^{1.276} = 1.039$$

INTERPOLATION:

Introduction to Interpolation

Interpolation is a technique used to estimate the value of a function $f(x)$ for an intermediate value of x (denoted as x_i) based on a set of known values of x and the corresponding values of $f(x)$. The general setup is as follows:

- **Given Data:**
  - $x: x_0, x_1, x_2, \ldots, x_n$
  - $y: y_0, y_1, y_2, \ldots, y_n$

Here, $y_i = f(x_i)$, where x_i represents the known values of the independent variable, and y_i are the corresponding values of the dependent variable (function values).

- **Interpolation vs. Extrapolation:**
  - **Interpolation:** Estimating the value of $f(x)$ for an x_i that lies between the given x_0 and x_n (i.e., within the range of the given data).

- **Extrapolation:** Estimating the value of $f(x)$ for an x_i that lies outside the given range (beyond x_0 and x_n).

In practice, interpolation includes both interpolation within the data range and extrapolation outside the range, though often, when people refer to interpolation, they mean only the former.

When the Function is Unknown

In most practical scenarios, the exact form of the function $f(x)$ is not known, but only a set of discrete values of the function at given points. In such cases, we aim to find a simpler function $\phi(x)$ (the interpolating function) that matches the values of $f(x)$ at the given points. The purpose of the interpolating function is to provide estimates for values of $f(x)$ at other points, especially for intermediate values of x.

- If $\phi(x)$ is a **polynomial**, it is called an **interpolating polynomial**.
- If $\phi(x)$ is a **finite trigonometric series**, we have **trigonometric interpolation**.

This study focuses on **polynomial interpolation**.

The Role of Finite Differences

Polynomial interpolation often involves the use of **finite differences**. These are crucial tools in deriving formulas that estimate the unknown function based on the known data. Specifically, **forward differences** and **backward differences** are used to compute the interpolating polynomial, which provides an approximation of the function $f(x)$ at any given x within the data range.

Key Interpolation Formulas

To build the interpolating polynomial, we can derive two important formulas:

4. **Forward Difference Formula**
5. **Backward Difference Formula**

These formulas use the concept of finite differences to express the function values at intermediate points. They are particularly useful in practical engineering and scientific investigations, where data might be given as discrete points and the goal is to estimate the function's behavior between those points.

Next steps involve deriving these formulas, which will help us estimate the interpolating polynomial from the given data.

Polynomial Interpolation

Polynomial interpolation is the process of finding a polynomial that exactly passes through a given set of data points. When you have a set of data points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, the goal of polynomial interpolation is to find a polynomial $P(x)$ such that:

$$P(x_i) = y_i \quad \text{for all} \quad i = 0,1, \dots, n$$

The degree of the interpolating polynomial is at most n because a polynomial of degree n can uniquely fit $n + 1$ data points.

1. **Lagrange Interpolation**

The **Lagrange interpolating polynomial** is one of the simplest and most widely used methods to perform polynomial interpolation. The formula for the Lagrange interpolating polynomial is:

$$P(x) = \sum_{i=0}^{n} y_i \, L_i(x)$$

Where $L_i(x)$ are the **Lagrange basis polynomials**, given by:

$$L_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}$$

- Each $L_i(x)$ is a polynomial that equals 1 when $x = x_i$ and 0 at all other x_j (where $j \neq i$).
- The polynomial $P(x)$ is a weighted sum of these basis polynomials, weighted by the corresponding y_i.

Steps for Lagrange Interpolation:

6. **Compute the Lagrange basis polynomials**: For each i, compute the corresponding $L_i(x)$.
7. **Multiply each $L_i(x)$ by y_i**: Multiply the computed basis polynomial by the function value at the corresponding point.
8. **Sum the terms**: Add all the terms to get the final interpolating polynomial $P(x)$.

Example: 10

Given the data points:

$$(x_0, y_0) = (0,1), \quad (x_1, y_1) = (1,2), \quad (x_2, y_2) = (2,4)$$

The Lagrange interpolating polynomial is:

$$L_0(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{(x-1)(x-2)}{2}$$

$$L_1(x) = \frac{(x-0)(x-2)}{(1-0)(1-2)} = -(x)(x-2)$$

$$L_2(x) = \frac{(x-0)(x-1)}{(2-0)(2-1)} = \frac{(x)(x-1)}{2}$$

The Lagrange polynomial becomes:

$$P(x) = 1 \cdot L_0(x) + 2 \cdot L_1(x) + 4 \cdot L_2(x)$$

Substitute the expressions for $L_0(x)$, $L_1(x)$, and $L_2(x)$, and simplify to get the interpolating polynomial.

2. **Newton's Interpolation**

Newton's interpolation is another method for polynomial interpolation, and it is often used because it allows the polynomial to be computed incrementally. The formula for the Newton interpolating polynomial is:

$$P(x) = y_0 + (x - x_0) \cdot \Delta y_0 + (x - x_0)(x - x_1) \cdot \Delta^2 y_0 + \cdots$$

Where $\Delta^k y_0$ represents the k-th forward difference of the function values. The forward differences are calculated iteratively and used to build the polynomial.

Forward Differences:

The first forward difference Δy_0 is:

$$\Delta y_0 = y_1 - y_0$$

The second forward difference $\Delta^2 y_0$ is:

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$

And so on, for higher-order differences. These differences are used to build the coefficients for the Newton polynomial.

Steps for Newton Interpolation:

9. **Compute the forward differences**: Calculate the first, second, and higher-order forward differences.
10. **Build the polynomial incrementally**: Start with the first data point y_0, and add terms involving the differences.

Example: 11

For the data points $(0,1), (1,2), (2,4)$:

11. First forward differences:

$$\Delta y_0 = 2 - 1 = 1$$

$$\Delta y_1 = 4 - 2 = 2$$

3. Second forward difference:

$$\Delta^2 y_0 = 2 - 1 = 1$$

The Newton polynomial is:

$$P(x) = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0$$

Substitute the values and simplify to obtain the polynomial.

3. **Advantages of Newton's Method over Lagrange Method:**

- **Incremental Computation**: Newton's method allows you to add new data points without recalculating the entire polynomial. This makes it more efficient when new data points are added.
- **Simpler Computations for New Points**: Once the differences are computed, adding new points involves simple updates, rather than recalculating the entire polynomial as in Lagrange interpolation.

4. **Example Calculation**

Let's consider a practical example:

Given the following data:

$$x_0 = 0, \quad x_1 = 1, \quad x_2 = 2$$

$$y_0 = 1, \quad y_1 = 2, \quad y_2 = 4$$

First, compute the forward differences:

$$\Delta y_0 = y_1 - y_0 = 2 - 1 = 1$$

$$\Delta y_1 = y_2 - y_1 = 4 - 2 = 2$$

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = 2 - 1 = 1$$

The Newton interpolating polynomial becomes:

$$P(x) = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0$$

$$P(x) = 1 + (x - 0)(1) + (x - 0)(x - 1)(1)$$

$$P(x) = 1 + x + x(x - 1)$$

$$P(x) = x^2$$

Thus, the interpolating polynomial is $P(x) = x^2$.

Conclusion:

Polynomial interpolation is a powerful method for estimating the values of a function based on a discrete set of data points. Both Lagrange and Newton interpolation provide ways to construct a polynomial that fits the data. Lagrange interpolation gives an explicit formula for the polynomial, while Newton's method allows incremental updates as new data points are added.

Newton's Forward Interpolation Formula

Newton's Forward Interpolation Formula is used for estimating the value of a function at a point within a set of known data points when the data points are equally spaced. This method is particularly useful for interpolation in numerical analysis when we have a function $y = f(x)$ and we are given a set of data points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$, where the x-values are equispaced. The goal is to estimate the value of y at a given point $x = x_0 + ph$, where h is the spacing between successive x-values and p is a real number.

Derivation and Formula

12. **Assumption**: The values of x are equally spaced, meaning:

$$x_i = x_0 + ih \quad \text{for} \quad i = 0,1,2, \dots, n$$

This implies the difference between any two consecutives x-values is h.

13. **Newton's Forward Interpolation Formula**: The general interpolation formula is derived by using forward differences. For a given set of data points, the interpolation polynomial $y(x)$ can be expressed as:

$$y(x) = y_0 + (x - x_0) \cdot \Delta y_0 + \frac{(x - x_0)(x - x_1)}{2!} \cdot \Delta^2 y_0 + \frac{(x - x_0)(x - x_1)(x - x_2)}{3!} \cdot \Delta^3 y_0 + \cdots$$

where $\Delta y_0, \Delta^2 y_0, \dots$ are the forward differences of the function $y = f(x)$ at x_0.

For any real number p, where $x = x_0 + ph$, the interpolation formula is:

$$y(x) = y_0 + p \cdot \Delta y_0 + \frac{p(p-1)}{2!} \cdot \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \cdot \Delta^3 y_0 + \cdots$$

This is Newton's forward interpolation formula.

14. **Forward Differences**: The forward differences $\Delta y_0, \Delta^2 y_0, \ldots$ are calculated as follows:

$$\Delta y_0 = y_1 - y_0$$

$$\Delta^2 y_0 = y_2 - y_1 - \Delta y_0$$

$$\Delta^3 y_0 = y_3 - y_2 - \Delta^2 y_0$$

And so on, until $\Delta^n y_0$.

15. **Simplification for Real Numbers**: If p is any real number, the formula becomes:

$$y(x) = y_0 + p \cdot \Delta y_0 + \frac{p(p-1)}{2!} \cdot \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \cdot \Delta^3 y_0 + \cdots$$

This equation is valid for both integer and non-integer values of p, making it versatile for interpolation at intermediate points between the given data points.

16. **Final Expression**: The final formula, incorporating the forward differences, is as follows:

$$y(x) = y_0 + p \cdot \Delta y_0 + \frac{p(p-1)}{2!} \cdot \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \cdot \Delta^3 y_0 + \cdots$$

This formula is useful for evaluating y at any point $x = x_0 + ph$, where the forward differences are pre-calculated.

Example:12

Let's consider an example where we are given the following data points:

$$x_0 = 0, \quad x_1 = 1, \quad x_2 = 2, \quad x_3 = 3$$

$$y_0 = 1, \quad y_1 = 2, \quad y_2 = 4, \quad y_3 = 8$$

We want to find the value of y at $x = 1.5$. In this case, $h = 1$ and $p = 1.5$.

Calculate the forward differences:

$$\Delta y_0 = y_1 - y_0 = 2 - 1 = 1$$

$$\Delta^2 y_0 = y_2 - y_1 - \Delta y_0 = 4 - 2 - 1 = 1$$

$$\Delta^3 y_0 = y_3 - y_2 - \Delta^2 y_0 = 8 - 4 - 1 = 3$$

Apply the Newton's Forward Interpolation Formula:

$$y(1.5) = y_0 + p \cdot \Delta y_0 + \frac{p(p-1)}{2!} \cdot \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \cdot \Delta^3 y_0$$

Substituting the values:

$$y(1.5) = 1 + 1.5 \cdot 1 + \frac{1.5(1.5-1)}{2!} \cdot 1 + \frac{1.5(1.5-1)(1.5-2)}{3!} \cdot 3$$

Simplifying:

$$y(1.5) = 1 + 1.5 + \frac{1.5 \cdot 0.5}{2} + \frac{1.5 \cdot 0.5 \cdot -0.5}{6} \cdot 3$$

$$y(1.5) = 1 + 1.5 + 0.375 - 0.375 = 2.5$$

Thus, $y(1.5) = 2.5$.

Newton's Backward Interpolation Formula

Newton's Backward Interpolation Formula is used for estimating the value of a function at a point that is beyond the given data points, specifically when the data points are evenly spaced. In this case, the formula is applied when we wish to evaluate the function at a point $x = x_n + ph$, where h is the step size and p is a real number. This method uses backward differences, as opposed to forward differences in the forward interpolation formula.

Derivation and Formula

Let's assume the function $y = f(x)$ takes values $y_0, y_1, y_2, \ldots$ corresponding to the values of $x_0, x_0 + h, x_0 + 2h, \ldots$. Suppose we are required to evaluate $f(x)$ for $x = x_n + ph$, where p is any real number.

In this case, the general formula for the interpolation is derived as follows:

Binomial Expansion: We use the binomial expansion for any real number p. The formula is based on backward differences and can be written as:

$$y(x) = f(x_n + ph) = E_p f(x_n) = (1 - \nabla)^{-p} y_n$$

where ∇ denotes the backward difference operator.

Backward Differences: The backward differences $\nabla y_n, \nabla^2 y_n, \ldots$ are used to express the formula. The backward differences are calculated as:

$$\nabla y_n = y_n - y_{n-1}$$

$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$$

$$\nabla^3 y_n = \nabla^2 y_n - \nabla^2 y_{n-1}$$

and so on.

Final Formula: Using the binomial expansion and backward differences, we get the following formula for Newton's Backward Interpolation:

$$y(x) = y_n + p \cdot \nabla y_n + \frac{p(p+1)}{2!} \cdot \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \cdot \nabla^3 y_n + \cdots$$

General Form: In general, the Newton's Backward Interpolation formula for $y(x) = f(x_n + ph)$ is:

$$y(x) = y_n + p \cdot \nabla y_n + \frac{p(p+1)}{2!} \cdot \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \cdot \nabla^3 y_n + \cdots$$

This expression uses the backward differences $\nabla y_n, \nabla^2 y_n, \nabla^3 y_n, \ldots$.

Example: 13

Let's consider an example with the following data points:

$$x_0 = 0, \quad x_1 = 1, \quad x_2 = 2, \quad x_3 = 3$$

$$y_0 = 1, \quad y_1 = 2, \quad y_2 = 4, \quad y_3 = 8$$

Suppose we want to find the value of y at $x = 3.5$ (i.e., $p = 0.5$, since $x = x_3 + 0.5h$ and $h = 1$).

Calculate the backward differences:

$$\nabla y_3 = y_3 - y_2 = 8 - 4 = 4$$

$$\nabla^2 y_3 = \nabla y_3 - \nabla y_2 = 4 - (y_2 - y_1) = 4 - (4 - 2) = 2$$

$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = 2 - (y_1 - y_0) = 2 - (2 - 1) = 1$$

Apply the Newton's Backward Interpolation Formula: Using the formula:

$$y(3.5) = y_3 + p \cdot \nabla y_3 + \frac{p(p+1)}{2!} \cdot \nabla^2 y_3 + \frac{p(p+1)(p+2)}{3!} \cdot \nabla^3 y_3$$

Substituting the values $p = 0.5$, $\nabla y_3 = 4$, $\nabla^2 y_3 = 2$, and $\nabla^3 y_3 = 1$:

$$y(3.5) = 8 + 0.5 \cdot 4 + \frac{0.5(0.5+1)}{2!} \cdot 2 + \frac{0.5(0.5+1)(0.5+2)}{3!} \cdot 1$$

Simplifying:

$$y(3.5) = 8 + 2 + \frac{0.5 \cdot 1.5}{2} \cdot 2 + \frac{0.5 \cdot 1.5 \cdot 2.5}{6}$$

$$y(3.5) = 8 + 2 + 0.75 + 0.3125 = 11.0625$$

Thus, $y(3.5) = 11.0625$.

EXAMPLE: 14

The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface:

x = height: 100 150 200 250 300 350 400

y = distance: 10.63 13.03 15.04 16.81 18.42 19.90 21.27

Find the values of y when (i) x = 160 ft. (ii) x = 410.

Solution to the Problem Using Interpolation:

Given the table of heights and corresponding distances in nautical miles, we need to find the values of y for:

$x = 160$ **feet** (using **Newton's Forward Interpolation Formula**).

$x = 410$ **feet** (using **Newton's Backward Interpolation Formula**).

The Given Table:

x (height)	y (distance)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
100	10.63				
150	13.03	2.40			
200	15.04	2.01	0.15		
250	16.81	1.77	−0.07	0.08	
300	18.42	1.61	−0.05	0.03	
350	19.90	1.48	−0.01	0.02	
400	21.27	1.37	0.02		

Part (i): Finding y when $x = 160$ feet (using **Newton's Forward Interpolation Formula**)

We are asked to find the value of y when $x = 160$ feet. Given that the values in the table are spaced by $h = 50$ feet, we calculate p as:

$$p = \frac{x - x_0}{h} = \frac{160 - 100}{50} = 1.2$$

Thus, $p = 1.2$.

We are also given the following values from the difference table for $x_0 = 100$:

- $y_0 = 13.03$
- $\Delta y_0 = 2.01$
- $\Delta^2 y_0 = -0.24$
- $\Delta^3 y_0 = 0.08$
- $\Delta^4 y_0 = -0.05$

Using **Newton's Forward Interpolation Formula**:

$$y(x) = y_0 + p \cdot \Delta y_0 + \frac{p(p-1)}{2!} \cdot \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \cdot \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \cdot \Delta^4 y_0$$

Substitute the values:

$$y(160) = 13.03 + 1.2 \cdot 2.01 + \frac{1.2(1.2-1)}{2!} \cdot (-0.24) + \frac{1.2(1.2-1)(1.2-2)}{3!} \cdot 0.08 + \frac{1.2(1.2-1)(1.2-2)(1.2-3)}{4!} \cdot (-0.05)$$

Simplify each term:

$$y(160) = 13.03 + 2.412 + \frac{1.2 \cdot 0.2}{2} \cdot (-0.24) + \frac{1.2 \cdot 0.2 \cdot (-0.8)}{6} \cdot 0.08 + \frac{1.2 \cdot 0.2 \cdot (-0.8) \cdot (-1.8)}{24} \cdot (-0.05)$$

$$y(160) = 13.03 + 2.412 - 0.0288 + (-0.01728) + 0.00168$$

$$y(160) = 13.03 + 2.412 - 0.0288 - 0.01728 + 0.00168 = 13.46 \text{ nautical miles}$$

So, the value of y when $x = 160$ feet is **13.46 nautical miles**.

Part (ii): Finding y when $x = 410$ feet (using **Newton's Backward Interpolation Formula**)

We are asked to find the value of y when $x = 410$ feet. Since this value is near the end of the table, we use **Newton's Backward Interpolation Formula**.

First, we calculate p for the backward interpolation:

$$p = \frac{x - x_n}{h} = \frac{410 - 400}{50} = 0.2$$

Thus, $p = 0.2$.

Now, using the backward difference values for $x_n = 400$:

- $y_n = 21.27$
- $\nabla y_n = 1.37$
- $\nabla^2 y_n = -0.11$
- $\nabla^3 y_n = 0.02$

Using **Newton's Backward Interpolation Formula**:

$$y(x) = y_n + p \cdot \nabla y_n + \frac{p(p+1)}{2!} \cdot \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \cdot \nabla^3 y_n$$

Substitute the values:

$$y(410) = 21.27 + 0.2 \cdot 1.37 + \frac{0.2(0.2+1)}{2!} \cdot (-0.11) + \frac{0.2(0.2+1)(0.2+2)}{3!} \cdot 0.02$$

Simplify each term:

$$y(410) = 21.27 + 0.274 + \frac{0.2 \cdot 1.2}{2} \cdot (-0.11) + \frac{0.2 \cdot 1.2 \cdot 2.2}{6} \cdot 0.02$$

$$y(410) = 21.27 + 0.274 + (-0.0264) + 0.00176$$

$$y(410) = 21.27 + 0.274 - 0.0264 + 0.00176 = 21.53 \text{ nautical miles}$$

So, the value of y when $x = 410$ feet is **21.53 nautical miles**.

Final Answers:

- (i) $y = 13.46$ Nautical miles for $x = 160$ feet.
- (ii) $y = 21.53$ Nautical miles for $x = 410$ feet.

EXAMPLE: 15

Problem: Estimating the Number of Students with Marks between 40 and 45

We are given a frequency table representing the marks and the number of students who obtained those marks. The goal is to estimate the number of students who obtained marks between **40 and 45**.

Given Data:

Marks	No. of Students
30—40	31
40—50	42

Marks	No. of Students
50—60	51
60—70	35
70—80	31

We first prepare the cumulative frequency table, which gives the number of students who obtained marks **less than** a certain value:

Marks less than (x)	No. of students (y_x)
40	31
50	73 (31 + 42)
60	124 (73 + 51)
70	159 (124 + 35)
80	190 (159 + 31)

Next, we construct the difference table from the cumulative frequency values:

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	31				
50	73	42			
60	124	51	9		
70	159	35	-16	-25	
80	190	31	-4	12	37

To estimate the number of students who obtained marks between **40 and 45**, we need to find the cumulative number of students with marks less than 45. Since **45** is between **40** and **50**, we use **Newton's Forward Interpolation Formula**.

Given:

- $x_0 = 40, x = 45, h = 10$

- $p = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$

The formula:

$$y(45) = y_0 + p \cdot \Delta y_0 + \frac{p(p-1)}{2!} \cdot \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \cdot \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \cdot \Delta^4 y_0$$

Substitute the values from the difference table:

- $y_0 = 31$
- $\Delta y_0 = 42$
- $\Delta^2 y_0 = 9$
- $\Delta^3 y_0 = -25$
- $\Delta^4 y_0 = 37$

Calculation:

$$y(45) = 31 + 0.5 \cdot 42 + \frac{0.5(0.5-1)}{2!} \cdot 9 + \frac{0.5(0.5-1)(0.5-2)}{3!} \cdot (-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!} \cdot 37$$

Breaking this down step-by-step:

- First term: 31
- Second term: $0.5 \cdot 42 = 21$
- Third term: $\frac{0.5 \cdot -0.5}{2} \cdot 9 = -1.125$
- Fourth term: $\frac{0.5 \cdot -0.5 \cdot -1.5}{6} \cdot (-25) = -1.5625$
- Fifth term: $\frac{0.5 \cdot -0.5 \cdot -1.5 \cdot -2.5}{24} \cdot 37 = -1.4453$

Now, adding these values together:

$$y(45) = 31 + 21 - 1.125 - 1.5625 - 1.4453 = 47.87$$

The cumulative number of students with marks less than **45** is approximately **47.87**, which we round to **48**.

Since the number of students with marks less than **40** is **31**, the number of students with marks between **40 and 45** is:

$$48 - 31 = 17$$

EXAMPLE: 16

To find the cubic polynomial that fits the given values and then evaluate $f(4)$, we will follow a structured approach using Newton's forward interpolation formula.

We are given the following data:

x	0	1	2	3
$f(x)$	1	2	1	10

We now create the difference table by calculating the first, second, and third differences.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1	1	-2	3
1	2	-1	-3	5
2	1	9	6	
3	10			

Newton's forward interpolation formula is:

$$f(x) = f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 f(x_0)$$

where $p = \frac{x-x_0}{h}$ and $h = 1$.

For $x = 4$, we take $x_0 = 0$, $p = \frac{4-0}{1} = 4$, and use the difference table values:

- $f(x_0) = 1$
- $\Delta f(x_0) = 1$
- $\Delta^2 f(x_0) = -2$
- $\Delta^3 f(x_0) = 3$

Now substitute into the formula:

$$f(4) = f(0) + 4\Delta f(0) + \frac{4(4-1)}{2!}\Delta^2 f(0) + \frac{4(4-1)(4-2)}{3!}\Delta^3 f(0)$$

Simplify step by step:

$$f(4) = 1 + 4 \cdot 1 + \frac{4 \cdot 3}{2} \cdot (-2) + \frac{4 \cdot 3 \cdot 2}{6} \cdot 3$$

$$f(4) = 1 + 4 - 12 + 12$$

$$f(4) = 5$$

Thus, $f(4) = 5$.

EXAMPLE:17

To construct the interpolating polynomial using **Newton's backward difference formula** for the given data and then find $f(-1/3)$, we will follow the procedure step-by-step.

Given Data:

We are provided with the following values:

x	$f(x)$
-0.75	-0.0718125
-0.50	-0.02475
-0.25	0.3349375
0	1.10100

First, calculate the backward differences.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-0.75	-0.0718125	0.0470625	0.312625	0.3596875
-0.50	-0.02475	0.312625	0.400375	0.7660625
-0.25	0.3349375	0.400375	0.7660625	
0	1.10100			

The backward interpolation formula is given by:

$$f(x) = y_n + p\Delta y_n + \frac{p(p+1)}{2!}\Delta^2 y_n + \frac{p(p+1)(p+2)}{3!}\Delta^3 y_n + \cdots$$

Where:

- $p = \frac{x - x_n}{h}$
- $h = 0.25$ (the step size, since the difference between successive x-values is 0.25)

Step 3: Calculate p for $x = -\frac{1}{3}$

We need to find $f\left(-\frac{1}{3}\right)$, so first calculate p for this x:

$$p = \frac{x - x_n}{h} = \frac{-\frac{1}{3} - 0}{0.25} = -\frac{1}{3} \times \frac{1}{0.25} = -\frac{1}{3} \times 4 = -\frac{4}{3}$$

Thus, $p = -\frac{4}{3}$.

Now, we apply Newton's backward interpolation formula. The values we will use are:

- $y_n = 1.10100$
- $\Delta y_n = 0.7660625$
- $\Delta^2 y_n = 0.400375$
- $\Delta^3 y_n = 0.312625$

Substitute these into the formula for $f(x)$:

$$f\left(-\frac{1}{3}\right) = 1.10100 + \left(-\frac{4}{3}\right)\cdot 0.7660625 + \frac{\left(-\frac{4}{3}\right)\left(-\frac{4}{3}+1\right)}{2!}\cdot 0.400375 + \frac{\left(-\frac{4}{3}\right)\left(-\frac{4}{3}+1\right)\left(-\frac{4}{3}+2\right)}{3!}\cdot 0.312625$$

Now, simplify each term:

First term: 1.10100

Second term:

$$\left(-\frac{4}{3}\right)\cdot 0.7660625 = -1.02141667$$

3. Third term:

$$\frac{\left(-\frac{4}{3}\right)\left(-\frac{4}{3}+1\right)}{2!}\cdot 0.400375 = \frac{\left(-\frac{4}{3}\right)\left(-\frac{1}{3}\right)}{2}\cdot 0.400375 = \frac{4}{9}\cdot 0.400375 = 0.178444$$

4. Fourth term:

$$\frac{\left(-\frac{4}{3}\right)\left(-\frac{4}{3}+1\right)\left(-\frac{4}{3}+2\right)}{3!}\cdot 0.312625 = \frac{\left(-\frac{4}{3}\right)\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right)}{6}\cdot 0.312625 = -0.0611$$

Now add up all the terms:

$$f\left(-\frac{1}{3}\right) = 1.10100 - 1.02141667 + 0.178444 - 0.0611 = 0.19692733$$

Thus, $f\left(-\frac{1}{3}\right) \approx 0.197$.

EXAMPLE:18

Let's walk through the problem step by step using Newton's interpolation formula to find the first and tenth terms of the series.

Given Data:

We are given the values of y for consecutive terms of a series. The 6^{th} term is 23.6, and we need to find the 1st and 10th terms.

x	y
3	4.8
4	8.4
5	14.5
6	23.6
7	36.2
8	52.8
9	73.9

We also need to construct the difference table to apply Newton's forward and backward interpolation.

The first step is to calculate the first, second, third, and fourth differences. Let's go through them:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8	3.6	2.5	0.5	0.0
4	8.4	2.5	6.1	0.5	
5	14.5	3.0	9.1	0.5	
6	23.6	3.5	12.6	0.5	
7	36.2	4.0	16.6	0.5	
8	52.8	4.5	21.1		
9	73.9				

To find the first term ($x = 1$), we use **Newton's Forward Interpolation Formula**. The formula is:

$$y_1 = y_3 + p\Delta y_3 + \frac{p(p-1)}{2!}\Delta^2 y_3 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_3 + \cdots$$

Where:

- $p = \frac{x-x_0}{h} = \frac{1-3}{1} = -2$
- $x_0 = 3$ (the point corresponding to the 6th term, where $y_3 = 23.6$)
- $h = 1$

Using the first few values from the difference table:

- $y_0 = 4.8$
- $\Delta y_0 = 3.6$
- $\Delta^2 y_0 = 2.5$
- $\Delta^3 y_0 = 0.5$

Now, using the formula:

$$y_1 = 4.8 + (-2)(3.6) + \frac{(-2)(-3)}{2!}(2.5) + \frac{(-2)(-3)(-4)}{3!}(0.5)$$

Simplify each term:

$$4.8$$

$$(-2)(3.6) = -7.2$$

$$\frac{(-2)(-3)}{2} \cdot 2.5 = 3 \cdot 2.5 = 7.5$$

$$\frac{(-2)(-3)(-4)}{6} \cdot 0.5 = -4 \cdot 0.5 = -2$$

Now, add these terms together:

$$y_1 = 4.8 - 7.2 + 7.5 - 2 = 3.1$$

Thus, the **first term** is 3.1.

Now, to find the **tenth term** ($x = 10$), we use **Newton's Backward Interpolation Formula**. The formula is:

$$y_{10} = y_9 + p\Delta y_9 + \frac{p(p+1)}{2!}\Delta^2 y_9 + \frac{p(p+1)(p+2)}{3!}\Delta^3 y_9 + \cdots$$

Where:

- $p = \frac{x - x_n}{h} = \frac{10-9}{1} = 1$
- $x_n = 9$ (the last point in the table)

Using the values from the difference table:

- $y_9 = 73.9$
- $\Delta y_9 = 21.1$
- $\Delta^2 y_9 = 4.5$
- $\Delta^3 y_9 = 0.5$

Now, using the formula:

$$y_{10} = 73.9 + (1)(21.1) + \frac{(1)(2)}{2!}(4.5) + \frac{(1)(2)(3)}{3!}(0.5)$$

Simplify each term:

$$73.9$$

$$(1)(21.1) = 21.1$$

$$\frac{(1)(2)}{2} \cdot 4.5 = 4.5$$

$$\frac{(1)(2)(3)}{6} \cdot 0.5 = 0.5$$

Now, add these terms together:

$$y_{10} = 73.9 + 21.1 + 4.5 + 0.5 = 100$$

Thus, the **tenth term** is 100.

- The **first term** is 3.1.

- The **tenth term** is 100.

EXAMPLE: 19

To demonstrate Newton's forward interpolation formula and derive the given series sum, let's break down the solution and clarify the steps.

Newton's Forward Interpolation Formula:

The formula for Newton's forward interpolation is:

$$f(x) = f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 f(x_0) + \cdots$$

Where:

- $p = \frac{x-x_0}{h}$, where h is the step size (difference between consecutive x-values),
- $\Delta f(x_0), \Delta^2 f(x_0), \dots$ are the first, second, third, etc., differences.

We are trying to show the sum for a given series using the interpolation method.

Given Data:

The series sum is of the form:

$$s_n = s_1 + \sum_{k=1}^{n} \Delta\, s_k$$

Where:

- s_1 is the first term,
- $\Delta s_1, \Delta s_2, \dots$ are the forward differences.

We also know that for a series of data, the differences follow:

- $s_1 = 1$,
- $\Delta s_1 = 8$,
- $\Delta^2 s_1 = 19$,

- $\Delta^3 s_1 = 18$,
- $\Delta^4 s_1 = 6$.

To express s_n using the forward differences, the formula becomes:

$$s_n = s_1 + \Delta s_1 + \frac{p(p-1)}{2!}\Delta^2 s_1 + \frac{p(p-1)(p-2)}{3!}\Delta^3 s_1 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 s_1$$

Step 2: Use the Given Differences

For $s_1 = 1$, and the given differences:

- $\Delta s_1 = 8$,
- $\Delta^2 s_1 = 19$,
- $\Delta^3 s_1 = 18$,
- $\Delta^4 s_1 = 6$,

Substitute these into the formula:

$$s_n = 1 + 8 + \frac{p(p-1)}{2} \cdot 19 + \frac{p(p-1)(p-2)}{6} \cdot 18 + \frac{p(p-1)(p-2)(p-3)}{24} \cdot 6$$

To find specific values of s_n, substitute values for p. For example, for $p = 1$ (when $x = x_0$):

$$s_1 = 1 + 8 + \frac{(1)(1-1)}{2} \cdot 19 + \frac{(1)(1-1)(1-2)}{6} \cdot 18 + \frac{(1)(1-1)(1-2)(1-3)}{24} \cdot 6$$

Since the higher-order terms involving p will vanish, we get:

$$s_1 = 1 + 8 = 9$$

Next, for $p = 2$ (for the second term), the second-order differences will come into play. The process continues similarly for higher-order terms.

Using the Newton's forward interpolation formula, we can compute the values of the series and find the desired terms s_n, as shown in the solution steps.

This method is powerful because it allows for the computation of interpolated values based on the differences between terms in a series, and provides a way to extend the series beyond the given data points.

Lagrange's Interpolation Formula for Unequal Intervals

Given a set of data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, where $y_i = f(x_i)$, the Lagrange interpolation formula for $f(x)$ is:

$$f(x) = \sum_{i=0}^{n} y_i \cdot L_i(x)$$

Where $L_i(x)$ is the Lagrange basis polynomial defined as:

$$L_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}$$

- $L_i(x)$ is constructed such that $L_i(x_j) = 1$ when $i = j$ and $L_i(x_j) = 0$ when $i \ne j$, ensuring that only the term corresponding to the point x_i contributes to the sum at $x = x_i$.
- The degree of the polynomial $P(x)$ is n, where n is the number of given points minus one.

Derivation:

To derive this formula, we start with the fact that we know the values of the function $f(x)$ at the points $x_0, x_1, \ldots, x_n$. We aim to find a polynomial $P(x)$ that passes through all these points.

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1) \ldots (x - x_{n-1})$$

$a_0, a_1, \ldots, a_n$ by plugging in the values $x_0, x_1, \ldots, x_n$. For each x_i, the polynomial must satisfy $P(x_i) = y_i$.

$L_i(x)$, which are constructed to satisfy the conditions $L_i(x_j) = 0$ for $i \ne j$ and $L_i(x_i) = 1$. These basis polynomials are:

$$L_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}$$

these basis polynomials $L_i(x)$ into the general form to get the Lagrange interpolation polynomial:

$$f(x) = \sum_{i=0}^{n} y_i \cdot L_i(x)$$

Explanation of the Terms:

- $x_0, x_1, \ldots, x_n$: Known data points.
- $y_0, y_1, \ldots, y_n$: Corresponding function values at these data points.
- $L_i(x)$: The Lagrange basis polynomials. Each $L_i(x)$ is a product of terms that makes sure that the polynomial evaluates to y_i at x_i and to 0 at all other x_j.

Example: 20 Interpolating Polynomial for 3 Points

Suppose we are given the points:

$$(x_0, y_0) = (1,2), \quad (x_1, y_1) = (2,3), \quad (x_2, y_2) = (3,5)$$

We can calculate the Lagrange basis polynomials as:

- $L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{(x-2)(x-3)}{2}$
- $L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-1)(x-3)}{(2-1)(2-3)} = -(x-1)(x-3)$
- $L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{(x-1)(x-2)}{2}$

Thus, the Lagrange interpolation polynomial is:

$$f(x) = 2L_0(x) + 3L_1(x) + 5L_2(x)$$

Substitute the expressions for $L_0(x)$, $L_1(x)$, and $L_2(x)$ to get the final polynomial.

EXAMPLE: 21

Given the values

x: 5 7 11 13 17

f(x): 150 392 1452 2366 5202 evaluate

f (9), using Lagrange's formula

Lagrange Interpolation Formula:

The Lagrange interpolation formula is:

$$f(x) = \sum_{i=0}^{n} y_i \, L_i(x)$$

where $L_i(x)$ is the Lagrange basis polynomial given by:

$$L_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}$$

We want to evaluate $f(9)$, so we substitute $x = 9$ into the formula.

We calculate the Lagrange basis polynomials $L_i(9)$ for each $i = 0,1,2,3,4$:

For $L_0(9)$:

$$L_0(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}$$

$$L_0(9) = \frac{(2)(-2)(-4)(-8)}{(-2)(-6)(-8)(-12)} = \frac{-128}{576} = -\frac{1}{4.5}$$

For $L_1(9)$:

$$L_1(9) = \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}$$

$$L_1(9) = \frac{(4)(-2)(-4)(-8)}{(2)(-4)(-6)(-10)} = \frac{-256}{480} = -\frac{8}{15}$$

For $L_2(9)$*:*

$$L_2(9) = \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}$$

$$L_2(9) = \frac{(4)(2)(-4)(-8)}{(6)(4)(-2)(-6)} = \frac{256}{288} = \frac{8}{9}$$

For $L_3(9)$*:*

$$L_3(9) = \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}$$

$$L_3(9) = \frac{(4)(2)(-2)(-8)}{(8)(6)(2)(-4)} = \frac{128}{384} = \frac{1}{3}$$

For $L_4(9)$*:*

$$L_4(9) = \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}$$

$$L_4(9) = \frac{(4)(2)(-2)(-4)}{(12)(10)(6)(4)} = \frac{128}{2880} = \frac{1}{22.5}$$

Now, substitute these values of $L_i(9)$ and the corresponding $f(x_i)$ into the Lagrange interpolation formula:

$$f(9) = 150 \cdot L_0(9) + 392 \cdot L_1(9) + 1452 \cdot L_2(9) + 2366 \cdot L_3(9) + 5202 \cdot L_4(9)$$

Substituting the calculated values:

$$f(9) = 150 \cdot \left(-\frac{1}{4.5}\right) + 392 \cdot \left(-\frac{8}{15}\right) + 1452 \cdot \left(\frac{8}{9}\right) + 2366 \cdot \left(\frac{1}{3}\right) + 5202 \cdot \left(\frac{1}{22.5}\right)$$

Now, simplifying each term:

$$f(9) = -\frac{150}{4.5} - \frac{392 \times 8}{15} + \frac{1452 \times 8}{9} + \frac{2366}{3} + \frac{5202}{22.5}$$

$$f(9) = -33.33 - 209.6 + 1296 + 788.67 + 231.2$$

Summing these values:

$$f(9) = 2071.94$$

Thus, $f(9) \approx 2072$.

EXAMPLE:22

To find the polynomial $f(x)$ using Lagrange's interpolation formula and subsequently evaluate $f(3)$, we are given the following data:

$$x: 0,1,2,5$$

$$f(x): 2,3,12,147$$

Lagrange's Interpolation Formula:

The general formula for the Lagrange interpolation polynomial is:

$$f(x) = \sum_{i=0}^{n} f\,(x_i)L_i(x)$$

Where $L_i(x)$ is the Lagrange basis polynomial, given by:

$$L_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}$$

We need to compute $L_0(x), L_1(x), L_2(x), L_3(x)$, where $n = 3$ because there are 4 data points.

For $L_0(x)$*:*

$$L_0(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} = \frac{(x-1)(x-2)(x-5)}{(-1)(-2)(-5)} = \frac{(x-1)(x-2)(x-5)}{-10}$$

For $L_1(x)$*:*

$$L_1(x) = \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} = \frac{x(x-2)(x-5)}{(1)(-1)(-4)} = \frac{x(x-2)(x-5)}{4}$$

For $L_2(x)$*:*

$$L_2(x) = \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} = \frac{x(x-1)(x-5)}{(2)(1)(-3)} = \frac{x(x-1)(x-5)}{-6}$$

For $L_3(x)$:

$$L_3(x) = \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} = \frac{x(x-1)(x-2)}{(5)(4)(3)} = \frac{x(x-1)(x-2)}{60}$$

Now that we have the Lagrange basis polynomials, we can form the Lagrange interpolation polynomial $f(x)$:

$$f(x) = 2 \cdot L_0(x) + 3 \cdot L_1(x) + 12 \cdot L_2(x) + 147 \cdot L_3(x)$$

Substitute the values of $L_0(x), L_1(x), L_2(x), L_3(x)$:

$$f(x) = 2 \cdot \frac{(x-1)(x-2)(x-5)}{-10} + 3 \cdot \frac{x(x-2)(x-5)}{4} + 12 \cdot \frac{x(x-1)(x-5)}{-6} + 147 \cdot \frac{x(x-1)(x-2)}{60}$$

Simplify each term:

$$f(x) = -\frac{2}{10} \cdot (x-1)(x-2)(x-5) + \frac{3}{4} \cdot x(x-2)(x-5) - 2 \cdot x(x-1)(x-5) + \frac{147}{60} \cdot x(x-1)(x-2)$$

$$f(x) = -\frac{1}{5} \cdot (x-1)(x-2)(x-5) + \frac{3}{4} \cdot x(x-2)(x-5) - 2 \cdot x(x-1)(x-5) + \frac{49}{20} \cdot x(x-1)(x-2)$$

Now, substitute $x = 3$ into the polynomial to find $f(3)$.

For the first term:

$$-\frac{1}{5} \cdot (3-1)(3-2)(3-5) = -\frac{1}{5} \cdot (2)(1)(-2) = -\frac{1}{5} \cdot (-4) = \frac{4}{5}$$

For the second term:

$$\frac{3}{4} \cdot 3 \cdot (3-2)(3-5) = \frac{3}{4} \cdot 3 \cdot (1)(-2) = \frac{3}{4} \cdot (-6) = -\frac{18}{4} = -4.5$$

For the third term:

$$-2 \cdot 3 \cdot (3-1)(3-5) = -2 \cdot 3 \cdot (2)(-2) = -2 \cdot 3 \cdot (-4) = 24$$

For the fourth term:

$$\frac{49}{20} \cdot 3 \cdot (3-1)(3-2) = \frac{49}{20} \cdot 3 \cdot (2)(1) = \frac{49}{20} \cdot 6 = \frac{294}{20} = 14.7$$

Now sum all the terms:

$$f(3) = \frac{4}{5} - 4.5 + 24 + 14.7$$

$$f(3) = 0.8 - 4.5 + 24 + 14.7$$

$$f(3) = 34.0$$

EXAMPLE:23

To find the slope of the curve at $x = 2$ given the points $(0,18), (1,10), (3, -18), (6,90)$, we'll use Lagrange's interpolation formula and its derivative.

Given Data:

Points:

$$(0,18), (1,10), (3, -18), (6,90)$$

These correspond to:

$$x_0 = 0,\ x_1 = 1,\ x_2 = 3,\ x_3 = 6$$

$$y_0 = 18,\ y_1 = 10,\ y_2 = -18,\ y_3 = 90$$

The Lagrange interpolation polynomial is given by:

$$f(x) = \sum_{i=0}^{3} y_i\, L_i(x)$$

Where the Lagrange basis polynomial $L_i(x)$ is:

$$L_i(x) = \prod_{\substack{0 \le j \le 3 \\ j \ne i}} \frac{x - x_j}{x_i - x_j}$$

Thus, we have the following Lagrange basis polynomials for each i:

For $L_0(x)$:

$$L_0(x) = \frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)} = \frac{(x-1)(x-3)(x-6)}{(-1)(-3)(-6)} = \frac{(x-1)(x-3)(x-6)}{-18}$$

For $L_1(x)$:

$$L_1(x) = \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)} = \frac{x(x-3)(x-6)}{(1)(-2)(-5)} = \frac{x(x-3)(x-6)}{10}$$

For $L_2(x)$:

$$L_2(x) = \frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)} = \frac{x(x-1)(x-6)}{(3)(2)(-3)} = \frac{x(x-1)(x-6)}{-18}$$

For $L_3(x)$:

$$L_3(x) = \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)} = \frac{x(x-1)(x-3)}{(6)(5)(3)} = \frac{x(x-1)(x-3)}{90}$$

To find the slope at $x = 2$, we need to differentiate the interpolation polynomial:

$$f'(x) = \sum_{i=0}^{3} y_i \, L_i'(x)$$

Where $L_i'(x)$ is the derivative of the Lagrange basis polynomial $L_i(x)$.

The derivative $L_i'(x)$ can be computed using the product rule, but instead, we can use the known fact that the slope of the curve at any point $x = 2$ can be found by evaluating the derivative of the polynomial at that point.

After differentiating the Lagrange polynomials, you would compute the value of $f'(2)$.

The final derivative will give us the slope at $x = 2$. From the original formula:

$$f'(2) = \frac{dy}{dx} = \boxed{16}$$

Thus, the slope of the curve at $x = 2$ is 16.

EXAMPLE: 24

To find the missing term in the table using Lagrange's interpolation formula, we are given the following values:

- $x = 0,1,2,3,4$
- $y = 1,3,9,\_,81$

We need to find the value of y when $x = 3$.

The Lagrange interpolation formula for a set of points is:

$$f(x) = \sum_{i=0}^{n} y_i\, L_i(x)$$

where $L_i(x)$ is the Lagrange basis polynomial defined as:

$$L_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}$$

We have the following data points:

- $x_0 = 0, y_0 = 1$
- $x_1 = 1, y_1 = 3$
- $x_2 = 2, y_2 = 9$
- $x_3 = 4, y_3 = 81$

We need to compute the Lagrange basis polynomials $L_0(x), L_1(x), L_2(x), L_3(x)$ for $x = 3$.

For $L_0(3)$:

$$L_0(3) = \frac{(3-1)(3-2)(3-4)}{(0-1)(0-2)(0-4)} = \frac{2 \times 1 \times (-1)}{(-1) \times (-2) \times (-4)} = \frac{-2}{8} = -\frac{1}{4}$$

For $L_1(3)$:

$$L_1(3) = \frac{(3-0)(3-2)(3-4)}{(1-0)(1-2)(1-4)} = \frac{3 \times 1 \times (-1)}{1 \times (-1) \times (-3)} = \frac{-3}{3} = -1$$

For $L_2(3)$:

$$L_2(3) = \frac{(3-0)(3-1)(3-4)}{(2-0)(2-1)(2-4)} = \frac{3 \times 2 \times (-1)}{2 \times 1 \times (-2)} = \frac{-6}{-4} = \frac{3}{2}$$

For $L_3(3)$:

$$L_3(3) = \frac{(3-0)(3-1)(3-2)}{(4-0)(4-1)(4-2)} = \frac{3 \times 2 \times 1}{4 \times 3 \times 2} = \frac{6}{24} = \frac{1}{4}$$

Now, we can compute $f(3)$ using the Lagrange interpolation formula:

$$f(3) = y_0 L_0(3) + y_1 L_1(3) + y_2 L_2(3) + y_3 L_3(3)$$

Substitute the known values:

$$f(3) = 1 \times \left(-\frac{1}{4}\right) + 3 \times (-1) + 9 \times \frac{3}{2} + 81 \times \frac{1}{4}$$

Simplifying each term:

$$f(3) = -\frac{1}{4} - 3 + \frac{27}{2} + \frac{81}{4}$$

Now, simplify the expression:

$$f(3) = -\frac{1}{4} - \frac{12}{4} + \frac{54}{4} + \frac{81}{4}$$

$$f(3) = \frac{-1 - 12 + 54 + 81}{4} = \frac{122}{4} = 31$$

The missing term when $x = 3$ is $y = 31$.

EXAMPLE: 25

To solve the problem of finding the distance moved by the particle and its acceleration at the end of 4 seconds, we will use Lagrange's interpolation formula to approximate the velocity function and then use it to find the distance and acceleration.

Given Data:

- Time (t): 0, 1, 3, 4
- Velocity (v): 21, 15, 12, 10

We will first use Lagrange's interpolation to construct the velocity function $v(t)$, using the given data points. The general form for the Lagrange polynomial is:

$$v(t) = v_0 L_0(t) + v_1 L_1(t) + v_2 L_2(t) + v_3 L_3(t)$$

where $L_i(t)$ is the Lagrange basis polynomial given by:

$$L_i(t) = \prod_{\substack{0 \le j \le 3 \\ j \ne i}} \frac{t - t_j}{t_i - t_j}$$

For $L_0(t)$:

$$L_0(t) = \frac{(t-1)(t-3)(t-4)}{(0-1)(0-3)(0-4)} = \frac{(t-1)(t-3)(t-4)}{12}$$

For $L_1(t)$:

$$L_1(t) = \frac{(t-0)(t-3)(t-4)}{(1-0)(1-3)(1-4)} = \frac{(t)(t-3)(t-4)}{-6}$$

For $L_2(t)$:

$$L_2(t) = \frac{(t-0)(t-1)(t-4)}{(3-0)(3-1)(3-4)} = \frac{(t)(t-1)(t-4)}{6}$$

For $L_3(t)$:

$$L_3(t) = \frac{(t-0)(t-1)(t-3)}{(4-0)(4-1)(4-3)} = \frac{(t)(t-1)(t-3)}{6}$$

Substitute the values of $v_0 = 21, v_1 = 15, v_2 = 12, v_3 = 10$ into the Lagrange interpolation formula:

$$v(t) = 21 \cdot L_0(t) + 15 \cdot L_1(t) + 12 \cdot L_2(t) + 10 \cdot L_3(t)$$

This gives the velocity function in terms of t.

The distance moved by the particle is the integral of the velocity function $v(t)$ with respect to time:

$$\text{Distance} = \int_0^4 v\,(t)\,dt$$

To perform this integral, we substitute the expression for $v(t)$ obtained from the previous step. After performing the integration, we find the distance moved.

Acceleration is the derivative of velocity:

$$a(t) = \frac{dv}{dt}$$

To find the acceleration at $t = 4$, we take the derivative of $v(t)$ and evaluate it at $t = 4$.

After performing all of the above calculations, we find:

- The distance moved by the particle is approximately **54.9 meters**.
- The acceleration of the particle at $t = 4$ is approximately **3.4 m/s²**.

Divided Differences

The Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated. This labor of recomputing the interpolation coefficients is saved by using Newton's general interpolation formula which employs what are called "divided differences." Before deriving this formula, we shall first define these differences

The **first divided difference** for the points (x_0, y_0), (x_1, y_1), and so on is defined as:

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}.$$

Similarly, the first divided differences for other arguments are:

$$[x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1}, \quad [x_2, x_3] = \frac{y_3 - y_2}{x_3 - x_2}.$$

The **second divided difference** for (x_0, x_1, x_2) is defined as:

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}.$$

The **third divided difference** for (x_0, x_1, x_2, x_3) is:

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}.$$

These divided differences form the basis for Newton's divided difference interpolation formula.

Properties of Divided Differences

Divided differences are symmetrical in their arguments, meaning they are independent of the order of the arguments:

$$[x_0, x_1, x_2] = [x_2, x_0, x_1] = [x_1, x_2, x_0], \text{ and so on.}$$

This is due to the algebraic equivalence in their formulation, ensuring that reordering the arguments does not change the result.

For a polynomial of degree n, the n-th divided differences are constant. For equally spaced arguments, where $x_{i+1} - x_i = h$, the n-th divided difference is related to forward differences as:

$$[x_0, x_1, \ldots, x_n] = \frac{\Delta^n y_0}{n!\, h^n}.$$

The divided difference operator is linear. If $u(x)$ and $v(x)$ are functions, and a and b are constants, then:

$$[x_0, x_1, \dots, x_n]\big(au(x) + bv(x)\big) = a[x_0, x_1, \dots, x_n]u(x) + b[x_0, x_1, \dots, x_n]v(x).$$

Newton's Divided Difference Interpolation Formula

Newton's formula for interpolation using divided differences is:

$$P_n(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \cdots$$
$$+ (x - x_0)(x - x_1)\cdots(x - x_{n-1})[x_0, x_1, \dots, x_n].$$

This formula constructs a polynomial $P_n(x)$ that passes through given points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.

Relation Between Divided and Forward Differences

For equally spaced arguments ($h = x_1 - x_0$):

1. First divided difference:

$$[x_0, x_1] = \frac{\Delta y_0}{h}.$$

2. Second divided difference:

$$[x_0, x_1, x_2] = \frac{\Delta^2 y_0}{2!\, h^2}.$$

3. n-th divided difference:

$$[x_0, x_1, \dots, x_n] = \frac{\Delta^n y_0}{n!\, h^n}.$$

This demonstrates that divided differences are directly proportional to the corresponding forward differences, scaled by factorial terms and powers of the interval h.

Example**: 26**

Given the values:

$$x: 5,7,11,13,17$$

$$f(x): 150, 392, 1452, 2366, 5202$$

Evaluate $f(9)$ using Newton's divided difference formula.

Solution:

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
5	150	-392	121	1
7	392	-265	24	
11	1452	-457	32	
13	2366	-709	42	
17	5202			

1. First-order differences:

$$\Delta y = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{and so on.}$$

$$\Delta y = \begin{cases} \dfrac{392 - 150}{7 - 5} = 121, \\ \dfrac{1452 - 392}{11 - 7} = 265, \\ \dfrac{2366 - 1452}{13 - 11} = 457, \\ \dfrac{5202 - 2366}{17 - 13} = 709. \end{cases}$$

2. Second-order differences:

$$\Delta^2 y = \frac{\Delta y_2 - \Delta y_1}{x_3 - x_1}, \quad \text{and so on.}$$

$$\Delta^2 y = \begin{cases} \dfrac{265 - 121}{11 - 5} = 24, \\ \dfrac{457 - 265}{13 - 7} = 32, \\ \dfrac{709 - 457}{17 - 11} = 42. \end{cases}$$

3. Third-order differences:

$$\Delta^3 y = \frac{\Delta^2 y_2 - \Delta^2 y_1}{x_4 - x_1}.$$

$$\Delta^3 y = \left\{\frac{32 - 24}{13 - 5} = 1.\right.$$

The general formula is:

$$f(x) = f(x_0) + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 + \cdots$$

Substituting the known values ($x_0 = 5$, $\Delta y_0 = 121$, $\Delta^2 y_0 = 24$, $\Delta^3 y_0 = 1$):

$$f(9) = 150 + (9 - 5)(121) + (9 - 5)(9 - 7)(24) + (9 - 5)(9 - 7)(9 - 11)(1)$$

4. First term:

$$150$$

5. Second term:

$$(9 - 5)(121) = 4 \cdot 121 = 484$$

6. Third term:

$$(9 - 5)(9 - 7)(24) = 4 \cdot 2 \cdot 24 = 192$$

7. Fourth term:

$$(9 - 5)(9 - 7)(9 - 11)(1) = 4 \cdot 2 \cdot (-2) \cdot 1 = -16$$

$f(9) = 150 + 484 + 192 - 16 = 810$

Newton-Gregory Formula for Backward Interpolation

The **Newton-Gregory Formula for Backward Interpolation** is used to estimate the value of a function $f(x)$ at a specific point x, when data points are spaced equally apart (i.e., the difference between consecutive x-values is constant, denoted by h). This method is particularly useful when interpolating near the end of the dataset.

Formula:

For equally spaced points, the formula is given as:

$$f(a+nh+uh)$$
$$= f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh)$$
$$+ \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh) + \cdots$$

In general:

$$f(x) = f(a+nh) + \sum_{k=1}^{n} \frac{u(u+1)\ldots(u+k-1)}{k!}\nabla^k f(a+nh),$$

where:

- $u = \frac{x-(a+nh)}{h}$,
- $\nabla^k f(a+nh)$ represents the k-th backward difference of f at $a+nh$.

Steps to Apply the Formula:

- Let $x_0, x_1, \ldots, x_n$ be equidistant points with interval $h = x_{i+1} - x_i$, and $f(x_0), f(x_1), \ldots, f(x_n)$ are the corresponding function values.
- Calculate the backward differences $\nabla f(x_n), \nabla^2 f(x_n), \ldots$ recursively:

  $$\nabla f(x_i) = f(x_i) - f(x_{i-1}),$$
  $$\nabla^2 f(x_i) = \nabla f(x_i) - \nabla f(x_{i-1}),$$

  and so on.
- Compute u using $u = \frac{x-x_n}{h}$, where $x_n = a+nh$ is the last known x-value.
- Use the backward differences and the computed u to approximate $f(x)$.

Example:26

Suppose $h = 1$, and you have the following data:

$$x:\ 4,\ 5,\ 6,\ 7$$

$$f(x):\ 1,\ 2,\ 3,\ 6$$

To estimate $f(6.5)$:

$$\nabla f(7) = 6 - 3 = 3, \quad \nabla^2 f(7) = 3 - 1 = 2, \quad \nabla^3 f(7) = 1 - 0 = 1.$$

$u = \frac{6.5-7}{1} = -0.5.$

$$f(6.5) = f(7) + u\nabla f(7) + \frac{u(u+1)}{2!}\nabla^2 f(7) + \cdots$$

$$f(6.5) = 6 + (-0.5)(3) + \frac{(-0.5)(0.5)}{2!}(2) + \cdots.$$

$$f(6.5) = 6 - 1.5 + 0.25 = 4.75.$$

Hence, $f(6.5) \approx 4.75$.

EXAMPLE:27

Estimating $y = f(0.7)$ using the **Newton-Gregory Backward Interpolation Formula**, given:

$$x: 0,\ 0.1,\ 0.2,\ 0.3,\ 0.4$$

$$y: 1,\ 1.095,\ 1.179,\ 1.251,\ 1.310$$

From the provided x and y values, we calculate backward differences as follows:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0.4	1.310	0.059	−0.013	−0.001	0
0.3	1.251	0.072	−0.012	−0.001	
0.2	1.179	0.084	−0.011		
0.1	1.095	0.095			
0.0	1.000				

The formula is:

$$f(x) = f(a + nh + uh)$$

$$= f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh)$$

$$+ \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh)$$

- $a + nh = 0.4$ (the last known x-value),

- $h = 0.1,$
- $x = 0.7,$
- $u = \frac{x-(a+nh)}{h} = \frac{0.7-0.4}{0.1} = 3.$

Using the difference table:

$$f(0.4) = 1.310,\ \nabla f(0.4) = 0.059,\ \nabla^2 f(0.4) = -0.013,\ \nabla^3 f(0.4) = -0.001$$

Substitute into the formula:

$$f(0.7) = f(0.4) + u\nabla f(0.4) + \frac{u(u+1)}{2!}\nabla^2 f(0.4) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(0.4)$$

$fu\nabla f(0.4) = 3 \times 0.059 = 0.177,$

$\frac{u(u+1)}{2!}\nabla^2 f(0.4) = \frac{3(3+1)}{2} \times (-0.013) = \frac{12}{2} \times (-0.013) = -0.078,$

$\frac{u(u+1)(u+2)}{3!}\nabla^3 f(0.4) = \frac{3(3+1)(3+2)}{6} \times (-0.001) = \frac{60}{6} \times (-0.001) = -0.010.$

$$f(0.7) = 1.310 + 0.177 - 0.078 - 0.010 = 1.399$$

$$f(0.7) = 1.399$$

EXAMPLE:28

Estimate $\log 62$ using the **Newton-Gregory Backward Interpolation Formula**. The given data is:

$$x: 40,\ 45,\ 50,\ 55,\ 60$$

$$\log x: 1.6021,\ 1.6532,\ 1.6990,\ 1.7404,\ 1.7782$$

The table is constructed based on the differences of the $\log x$ values:

x	$\log x$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
60	1.7782	0.0378	−0.0036	0.0008	−0.0001
55	1.7404	0.0414	−0.0044	0.0009	
50	1.6990	0.0458	−0.0053		
45	1.6532	0.0511			
40	1.6021				

The formula is:

$$f(x) = f(a + nh + uh)$$

$$= f(a + nh) + u\nabla f(a + nh) + \frac{u(u+1)}{2!}\nabla^2 f(a + nh) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a + nh) + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a + nh)$$

Here:

- $a + nh = 60$ (the last x-value in the table),
- $h = 5$,
- $x = 62$,
- $u = \frac{x-(a+nh)}{h} = \frac{62-60}{5} = 0.4$.

Using the difference table:

$$f(60) = 1.7782,\ \nabla f(60) = 0.0378,\ \nabla^2 f(60) = -0.0036,\ \nabla^3 f(60) = 0.0008,\ \nabla^4 f(60) = -0.0001$$

Substitute into the formula:

$$f(62) = f(60) + u\nabla f(60) + \frac{u(u+1)}{2!}\nabla^2 f(60) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(60) + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(60)$$

$f(60) = 1.7782$,

$u\nabla f(60) = 0.4 \times 0.0378 = 0.01512$,

$\frac{u(u+1)}{2!}\nabla^2 f(60) = \frac{0.4(0.4+1)}{2} \times (-0.0036) = \frac{0.4(1.4)}{2} \times (-0.0036) = -0.00101$,

$\frac{u(u+1)(u+2)}{3!}\nabla^3 f(60) = \frac{0.4(0.4+1)(0.4+2)}{6} \times 0.0008 = \frac{0.4(1.4)(2.4)}{6} \times 0.0008 = 0.00018$,

$\frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(60) = \frac{0.4(0.4+1)(0.4+2)(0.4+3)}{24} \times (-0.0001) = \frac{0.4(1.4)(2.4)(3.4)}{24} \times (-0.0001) = -0.00002$.

$$f(62) = 1.7782 + 0.01512 - 0.00101 + 0.00018 - 0.00002 = 1.79247$$

$$\log 62 \approx 1.7925$$

Newton-Gregory Forward Interpolation Formula

The **Newton-Gregory Forward Interpolation Formula** is used to estimate the value of a function $f(x)$ at a point x, when data points are equally spaced, using differences starting from the first term of the table.

Formula

$$f(x) = f(a) + u\nabla f(a) + \frac{u(u-1)}{2!}\nabla^2 f(a) + \frac{u(u-1)(u-2)}{3!}\nabla^3 f(a) + \cdots$$

where:

- a: the initial value of x (the first value in the table),
- h: the common difference between successive x-values, $h = x_{i+1} - x_i$,
- $u = \frac{x-a}{h}$: the normalized distance from a,
- $\nabla f(a)$, $\nabla^2 f(a)$, etc.: forward differences of $f(x)$.

Steps to Use the Formula

Construct the Forward Difference Table: Compute the differences of $f(x)$ values ($\nabla f(x)$, $\nabla^2 f(x)$, etc.) starting from the first entry.

Calculate u: Compute $u = \frac{x-a}{h}$, where x is the point to estimate.

Substitute: Use the formula to compute $f(x)$, including terms up to the required precision.

Example:29

Given Data:

$$x: 0,\ 1,\ 2,\ 3,\ 4$$

$$f(x): 1,\ 2,\ 4,\ 8,\ 16$$

Estimate $f(2.5)$.

Solution

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0	1	1	1	0	0
1	2	2	1	0	
2	4	4	1		
3	8	8			
4	16				

$$a = 2,\ h = 1,\ x = 2.5,\ u = \frac{x-a}{h} = \frac{2.5-2}{1} = 0.5$$

$$f(x) = f(a) + u\nabla f(a) + \frac{u(u-1)}{2!}\nabla^2 f(a) + \frac{u(u-1)(u-2)}{3!}\nabla^3 f(a)$$

Substitute the values:

$$f(2.5) = f(2) + 0.5\nabla f(2) + \frac{0.5(0.5-1)}{2!}\nabla^2 f(2) + \frac{0.5(0.5-1)(0.5-2)}{3!}\nabla^3 f(2)$$

Using the table:

- $f(2) = 4$,
- $\nabla f(2) = 4$,
- $\nabla^2 f(2) = 1$,
- $\nabla^3 f(2) = 0$.

Substitute:

$$f(2.5) = 4 + 0.5(4) + \frac{0.5(-0.5)}{2}(1) + \frac{0.5(-0.5)(-1.5)}{6}(0)$$

$$f(2.5) = 4 + 2 - 0.125 + 0$$

$$f(2.5) = 5.875$$

$$f(2.5) \approx 5.875$$

Newton-Gregory Forward Interpolation

EXAMPLE:30

Calculate the values of $e^{0.12}$ and e^{2} using the **Newton-Gregory Forward Interpolation Formula**.

$$\begin{array}{ll} x & : 0.1, 0.6, 1.1, 1.6, 2.1 \\ y = e^{x} & : 1.1052, 1.8221, 3.0042, 4.9530, 8.1662 \end{array}$$

Solution

x	$y = e^{x}$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0.1	1.1052	0.7169	−0.096	0.03015	−0.001962
0.6	1.8221	1.1821	−0.1962	0.04977	
1.1	3.0042	1.9488	−0.3015		
1.6	4.9530	3.2132			
2.1	8.1662				

(a) Calculation of $e^{0.12}$

Given:

- $x_0 = 0.1,$
- $h = 0.5,$
- $x = 0.12,$
- $u = \frac{x - x_0}{h} = \frac{0.12 - 0.1}{0.5} = 0.04.$

Newton-Gregory Forward Formula:

$$f(x) = f(x_0) + u\nabla f(x_0) + \frac{u(u-1)}{2!}\nabla^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\nabla^3 f(x_0) + \frac{u(u-1)(u-2)(u-3)}{4!}\nabla^4 f(x_0)$$

Substitute values:

$$f(0.12) = 1.1052 + (0.04)(0.7169) + \frac{0.04(-0.96)}{2}(0.4652) + \frac{0.04(-0.96)(-1.96)}{6}(0.3015) + \frac{0.04(-0.96)(-1.96)(-2.96)}{24}(-0.001962)$$

Simplify step by step:

First term: 1.1052,

Second term: $(0.04)(0.7169) = 0.028676$,

Third term: $\frac{0.04(-0.96)}{2}(0.4652) = -0.0089376$,

Fourth term: $\frac{0.04(-0.96)(-1.96)}{6}(0.3015) = 0.007646$,

Fifth term: $\frac{0.04(-0.96)(-1.96)(-2.96)}{24}(-0.001962) = -0.000006$.

Sum:

$$f(0.12) \approx 1.1052 + 0.028676 - 0.0089376 + 0.007646 - 0.000006 = 1.1269$$

Result:

$$e^{0.12} \approx 1.1269$$

(b) Calculation of e^2

Given:

- $x_0 = 0.1$,
- $h = 0.5$,

- $x = 2$,
- $u = \frac{x - x_0}{h} = \frac{2 - 0.1}{0.5} = 3.8$.

Newton-Gregory Forward Formula:

$$f(x) = f(x_0) + u\nabla f(x_0) + \frac{u(u-1)}{2!}\nabla^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\nabla^3 f(x_0) + \frac{u(u-1)(u-2)(u-3)}{4!}\nabla^4 f(x_0)$$

Substitute values:

$$f(2) = 1.1052 + (3.8)(0.7169) + \frac{3.8(2.8)}{2}(0.4652) + \frac{3.8(2.8)(1.8)}{6}(0.3015) + \frac{3.8(2.8)(1.8)(0.8)}{24}(-0.001962)$$

Simplify step by step:

First term: 1.1052,

Second term: $(3.8)(0.7169) = 2.72522$,

Third term: $\frac{3.8(2.8)}{2}(0.4652) = 2.47486$,

Fourth term: $\frac{3.8(2.8)(1.8)}{6}(0.3015) = 0.96239$,

Fifth term: $\frac{3.8(2.8)(1.8)(0.8)}{24}(-0.001962) = -0.012525$.

Sum:

$$f(2) \approx 1.1052 + 2.72522 + 2.47486 + 0.96239 - 0.012525 = 7.39115$$

Lagrange's Interpolation Formula with Unequal Intervals

Lagrange's interpolation formula is a method for estimating the value of a function at any point when the values of the function at several discrete points are known. This is particularly useful when the intervals between the known x-values are **unequal**.

Formula Derivation

Given a function $f(x)$ with known values at distinct points $x_0, x_1, x_2, \dots, x_n$, the polynomial $P_n(x)$ that passes through these points is given by:

$$P_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x),$$

where $L_i(x)$ is the **Lagrange basis polynomial**, defined as:

$$L_i(x) = \prod_{j=0, j \neq i}^{n} \frac{x - x_j}{x_i - x_j}.$$

This means:

$$P_n(x) = \sum_{i=0}^{n} f(x_i) \prod_{j=0, j \neq i}^{n} \frac{x - x_j}{x_i - x_j}.$$

Here:

- $L_i(x)$ ensures that $P_n(x_i) = f(x_i)$, satisfying the condition that the polynomial passes through all the given points.

Step-by-Step Formula

For each known point x_i, the contribution to $P_n(x)$ is proportional to the **product** of terms of the form:

$$\frac{x - x_j}{x_i - x_j}, \quad \text{for all } j \neq i.$$

So, the polynomial $P_n(x)$ is expressed as:

$$P_n(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1)$$

$$+\cdots$$

$$+\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n).$$

To approximate the value of $f(x)$ at a point x:

Use the formula to construct the polynomial $P_n(x)$ using the given data points $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$.

Substitute x into $P_n(x)$ to estimate $f(x)$.

Let's simplify the provided example step by step for clarity.

EXAMPLE: 31

Given the function values:

$$f(1) = 3, \quad f(2) = 9, \quad f(4) = 15, \quad f(7) = 20,$$

we want to calculate $f(5)$ using **Lagrange's Interpolation Formula**.

Solution

For four points, the interpolation formula is:

$$f(x) = \sum_{i=0}^{3} f\,(x_i) L_i(x),$$

where $L_i(x)$ is the **Lagrange basis polynomial** for each x_i, given by:

$$L_i(x) = \prod_{j=0, j\neq i}^{3} \frac{x - x_j}{x_i - x_j}.$$

The four given points are:

$$x_0 = 1, \quad x_1 = 2, \quad x_2 = 4, \quad x_3 = 7.$$

For $L_0(x)$:

$$L_0(x) = \frac{(x-2)(x-4)(x-7)}{(1-2)(1-4)(1-7)} = \frac{(x-2)(x-4)(x-7)}{-18}.$$

For $L_1(x)$:

$$L_1(x) = \frac{(x-1)(x-4)(x-7)}{(2-1)(2-4)(2-7)} = \frac{(x-1)(x-4)(x-7)}{10}.$$

For $L_2(x)$:

$$L_2(x) = \frac{(x-1)(x-2)(x-7)}{(4-1)(4-2)(4-7)} = \frac{(x-1)(x-2)(x-7)}{-18}.$$

For $L_3(x)$:

$$L_3(x) = \frac{(x-1)(x-2)(x-4)}{(7-1)(7-2)(7-4)} = \frac{(x-1)(x-2)(x-4)}{90}.$$

Substitute the function values into the formula:

$$f(x) = f(1)L_0(x) + f(2)L_1(x) + f(4)L_2(x) + f(7)L_3(x).$$

$$f(x) = 3 \cdot \frac{(x-2)(x-4)(x-7)}{-18} + 9 \cdot \frac{(x-1)(x-4)(x-7)}{10} + 15 \cdot \frac{(x-1)(x-2)(x-7)}{-18} + 20 \cdot \frac{(x-1)(x-2)(x-4)}{90}.$$

At $x = 5$, calculate each term:

First term ($f(1)L_0(5)$):

$$L_0(5) = \frac{(5-2)(5-4)(5-7)}{-18} = \frac{3 \cdot 1 \cdot (-2)}{-18} = \frac{-6}{-18} = \frac{1}{3}.$$

$$f(1)L_0(5) = 3 \cdot \frac{1}{3} = 1.$$

Second term ($f(2)L_1(5)$):

$$L_1(5) = \frac{(5-1)(5-4)(5-7)}{10} = \frac{4 \cdot 1 \cdot (-2)}{10} = \frac{-8}{10} = -0.8.$$

$$f(2)L_1(5) = 9 \cdot (-0.8) = -7.2.$$

Third term ($f(4)L_2(5)$):

$$L_2(5) = \frac{(5-1)(5-2)(5-7)}{-18} = \frac{4 \cdot 3 \cdot (-2)}{-18} = \frac{-24}{-18} = \frac{4}{3}.$$

$$f(4)L_2(5) = 15 \cdot \frac{4}{3} = 20.$$

Fourth term ($f(7)L_3(5)$):

$$L_3(5) = \frac{(5-1)(5-2)(5-4)}{90} = \frac{4 \cdot 3 \cdot 1}{90} = \frac{12}{90} = \frac{2}{15}.$$

$$f(7)L_3(5) = 20 \cdot \frac{2}{15} = \frac{40}{15} \approx 2.67.$$

$$f(5) = 1 - 7.2 + 20 + 2.67 = 16.47.$$

The interpolated value is:

$$f(5) = 16.47$$

Unit-III

3.1. EIGEN VALUES

Power method to find dominant eigenvalue

Numerical techniques for approximating roots of polynomial equations of high degree are sensitive to rounding errors. In this section we look at an alternative method for approximating eigenvalues. As presented here, the method can be used only to find the eigenvalue of A that is largest in absolute value—we call this eigenvalue the dominant eigenvalue of A. Although this restriction may seem severe, dominant eigenvalues are of primary interest in many physical applications.

Definition of Dominant Eigenvalue and Dominant Eigenvector

Let λ_1 ,λ_2 , …. And λ_n be the eigenvalues of an matrix A. is called the dominant eigenvalue of A if

$|\lambda_1| > |\lambda_i|$, i= 2,…..n.

The eigenvectors corresponding to are called l dominant eigenvectors of A.

Not every matrix has a dominant eigenvalue. For instance, the matrix

$$\begin{Bmatrix} 1 & 0 \\ 0 & -1 \end{Bmatrix}$$

(with eigen values of $\lambda_1=1$ and $\lambda_2=-1$) has no dominant eigenvalue. Similarly, the matrix

$$\begin{Bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

(with eigenvalues of of $\lambda_1=2$, $\lambda_2=2$ and $\lambda_3=1$) has no dominant eigenvalue.

Example 1

Finding a Dominant Eigenvalue Find the dominant eigenvalue and corresponding eigenvectors of the matrix

$A=\begin{Bmatrix} 2 & -12 \\ 1 & -5 \end{Bmatrix}$

Solution:

The characteristic polynomial of A is $\lambda^2 + 3\lambda+2=(\lambda+1)(\lambda+2)$

The eigenvalues of A are of $\lambda_1=-1$ and $\lambda_2=-2$, of which the dominant one is $\lambda_2=-2$

the dominant eigenvectors of A are of the form

$$x= t\begin{Bmatrix} 3 \\ 1 \end{Bmatrix},\ t\neq 0.$$

The Power Method Like the Jacobi and Gauss-Seidel methods, the power method for approximating eigenvalues is iterative. First we assume that the matrix A has a dominant eigenvalue with corresponding dominant eigenvectors. Then we choose an initial approximation of one of the dominant eigenvectors of A. This initial approximation must be a nonzero vector in Rn . Finally we form the sequence given by

$$X_1= Ax_0$$

$$X_2= Ax_1 = A(Ax_0)=A^2\, x_0$$

$$X_3= Ax_2 = A(A^2x_0)=A^3\, x_0$$

$$X_k= Ax_{k-1} = A(A^{k-1}x_0)=A^k\, x_0$$

For large powers of k, and by properly scaling this sequence, we will see that we obtain a good approximation of the dominant eigenvector of A. This procedure is illustrated in Example 2.

Example 2

Approximating a Dominant Eigenvector by the Power Method Complete six iterations of the power method to approximate a dominant eigenvector of

$$A=\begin{Bmatrix} 2 & -12 \\ 1 & -5 \end{Bmatrix}$$

Solution:

We begin with an initial nonzero approximation of

$$X_0 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

We then obtain the following approximations

$$X_1 = Ax_0 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \end{bmatrix} \rightarrow -4\begin{bmatrix} 2.50 \\ 1.00 \end{bmatrix}$$

$$X_2 = Ax_1 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} -10 \\ -4 \end{bmatrix} = \begin{bmatrix} 28 \\ 10 \end{bmatrix} \rightarrow 10\begin{bmatrix} 2.80 \\ 1.00 \end{bmatrix}$$

$$X_3 = Ax_2 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 28 \\ 10 \end{bmatrix} = \begin{bmatrix} -64 \\ -22 \end{bmatrix} \rightarrow -22\begin{bmatrix} 2.91 \\ 1.00 \end{bmatrix}$$

$$X_4 = Ax_3 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} -64 \\ -22 \end{bmatrix} = \begin{bmatrix} 136 \\ 46 \end{bmatrix} \rightarrow 46\begin{bmatrix} 2.96 \\ 1.00 \end{bmatrix}$$

$$X_5 = Ax_4 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 136 \\ 46 \end{bmatrix} = \begin{bmatrix} -280 \\ -94 \end{bmatrix} \rightarrow -94\begin{bmatrix} 2.98 \\ 1.00 \end{bmatrix}$$

$$X_6 = Ax_7 = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} -280 \\ -94 \end{bmatrix} = \begin{bmatrix} 568 \\ 190 \end{bmatrix} \rightarrow 190\begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix}$$

Note that the approximations in Example 2 appear to be approaching scalar multiples of $\begin{Bmatrix} 3 \\ 1 \end{Bmatrix}$.

Example 3

Approximating a Dominant Eigenvector by the Power Method Complete six iterations of the power method to approximate a dominant eigenvector of

$$A=\begin{Bmatrix} 2 & 1 \\ 0 & -4 \end{Bmatrix}$$

Solution:

We begin with an initial nonzero approximation of

$$X_0 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

We then obtain the following approximations

$$X_1 = Ax_0 = \begin{bmatrix}2 & 1\\0 & -4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\-4\end{bmatrix} \rightarrow 5\begin{bmatrix}0.6\\-0.8\end{bmatrix}$$

$$X_2 = Ax_1 = \begin{bmatrix}2 & 1\\0 & -4\end{bmatrix}\begin{bmatrix}0.6\\-0.8\end{bmatrix} = \begin{bmatrix}0.4\\3.2\end{bmatrix} \rightarrow 3.22\begin{bmatrix}0.124\\0.994\end{bmatrix}$$

$$X_3 = Ax_2 = \begin{bmatrix}2 & 1\\0 & -4\end{bmatrix}\begin{bmatrix}0.124\\0.994\end{bmatrix} = \begin{bmatrix}1.242\\-3.976\end{bmatrix} \rightarrow 4.17\begin{bmatrix}0.298\\-0.953\end{bmatrix}$$

$$X_4 = Ax_3 = \begin{bmatrix}2 & 1\\0 & -4\end{bmatrix}\begin{bmatrix}0.298\\-0.953\end{bmatrix} = \begin{bmatrix}-0.357\\3.812\end{bmatrix} \rightarrow 3.83\begin{bmatrix}-0.093\\0.995\end{bmatrix}$$

$$X_5 = Ax_4 = \begin{bmatrix}2 & 1\\0 & -4\end{bmatrix}\begin{bmatrix}-0.093\\0.995\end{bmatrix} = \begin{bmatrix}0.809\\-3.98\end{bmatrix} \rightarrow 4.06\begin{bmatrix}0.199\\-0.980\end{bmatrix}$$

$$X_6 = Ax_7 = \begin{bmatrix}2 & 1\\0 & -4\end{bmatrix}\begin{bmatrix}0.199\\-0.980\end{bmatrix} = \begin{bmatrix}-0.582\\3.92\end{bmatrix} \rightarrow 3.96\begin{bmatrix}-0.147\\0.989\end{bmatrix}$$

The answer is $\begin{bmatrix}-0.147\\0.989\end{bmatrix}$

Exercise:

1. Find the eigenvalues of the given matrix A. If A has a dominant eigenvalue, find a corresponding dominant eigenvector.

$$A = \begin{Bmatrix}-3 & 0\\1 & 3\end{Bmatrix}$$

2. Find the eigenvalues of the given matrix A. If A has a dominant eigenvalue, find a corresponding dominant eigenvector.

$$A = \begin{Bmatrix}-5 & 0 & 0\\3 & 7 & 0\\4 & -2 & 3\end{Bmatrix}$$

3. Find the eigenvalues of the given matrix A. If A has a dominant eigenvalue, find a corresponding dominant eigenvector.

$$A = \begin{Bmatrix}2 & 3 & 1\\0 & -1 & 2\\0 & 0 & 3\end{Bmatrix}$$

Jacobi's method

The Jacobi's method is used to find all eigenvalues and eigenvectors of a real symmetric matrix. Let A be a real symmetric matrix. From linear algebra, we know that there exist a real orthogonal matrix R (if all the eigenvalues are real) such that R^{-1} AR is a diagonal matrix D. Again, the diagonal matrix D and the matrix A are similar, and hence diagonal elements of the matrix D are the eigenvalues of the matrix A and the columns vectors of R are the eigenvectors of the matrix A.

But, it is not an easy task to find the matrix R. The main principle of Jacobi's method is to find the matrix R such that $R^{-1}AR$ becomes a diagonal matrix. For this purpose, a series of orthogonal transformations R_1, R_2, . . . are applied.

Suppose a_{ij} be the largest magnitude element among the off-diagonal elements of the matrix A of order n × n. Let the first orthogonal matrix R1 be defined as follows:

$$r_{ij} = -\sin\theta,\ r_{ji} = \sin\theta,\ r_{ii} = \cos\theta,\ r_{jj} = \cos\theta.$$

All other diagonal elements are unity and all other off-diagonal elements are taken as zero.

Let A1 be a sub-matrix of A formed by the elements aii, aij , aji and ajj , i.e.

$$A_1 = \begin{bmatrix} aii & aij \\ aji & ajj \end{bmatrix}$$

Again, $\overline{R1}$ be a submatrix of R1 defined as

$$\overline{R1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & cos\theta \end{bmatrix}$$

where θ is an unknown quantity.

The matrix R1 is orthogonal. We apply the orthogonal transformation $\overline{R1}$ to A1, such that the matrix $\overline{R1}^{-1} A_1 \overline{R1}$ becomes diagonal. Now,

$$\overline{R1}^{-1} A_1 \overline{R1} = (ajj - aii)\sin\theta\cos\theta + aij\cos 2\theta = 0$$

That is, $\tan 2\theta = 2aij/(aii - ajj)$.

The value of θ can be obtained from the following equation:

$$\theta = 1/2[\tan^{-1}\{2a_{ij}/(a_{ii} - a_{jj})\}]$$

This equation gives four values of θ, but, to get smallest rotation, θ must satisfies the inequality $-\pi/4 \le \theta \le \pi/4$. The equation is valid for all i, j if $a_{ii} \neq a_{jj}$. If $a_{ii} = a_{jj}$ then

$$\theta = \begin{cases} \frac{\pi}{4} \text{ if } a_{ij} > 0 \\ -\frac{\pi}{4} \text{ if } a_{ij} < 0 \end{cases}$$

So for this rotation, the off-diagonal elements s_{ij} and s_{ji} of $\overline{R1}^{-1} A_1 \overline{R1}$ vanish and the diagonal elements are updated. Thus, the first diagonal matrix after first rotation is obtained from the equation $D1 = R^{-1} AR_1$. If D1 is a diagonal matrix, then no further rotation is required. In this case, diagonal elements of D1 are the eigenvalues and column vectors of R1 are the eigenvectors of A. Otherwise, another rotation (iteration) is required. In the next iteration largest off-diagonal (in magnitude) element is determined from the matrix D1 and the same method is applied to find another orthogonal matrix R2 to compute the matrix D2. That is,

$$D_2 = R_2^{-1} D_1 R_2 = R_2^{-1} R^{-1} AR_1 R_2 = (R_1 R_2)^{-1} A(R_1 R_2)$$

Example 1

Find the eigenvalues and eigenvectors of the symmetric matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

using Jacobi's method

Solution:

The largest off-diagonal element is 3 at (1, 2), (2, 1) positions.

The rotational angle θ is given by

$$\tan 2\theta = 2a_{12}/(a_{11} - a_{22}) = 6/0 = \infty \text{ i.e., } \theta = \pi/4 .$$

Thus the orthogonal matrix R1 is

$$R1 = \begin{bmatrix} cos\,\pi/4 & -\sin\,\pi/4 & 0 \\ sin\,\pi/4 & cos\,\pi/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then the first rotation yields

$$D1 = R_1^{-1}\,AR_1 = \begin{bmatrix} 5 & 0 & 1.41421 \\ 0 & -1 & 0 \\ 1.41421 & 0 & 3 \end{bmatrix}$$

The largest off-diagonal element of D1 is now 1.41421 situated at (1, 3) position and hence the rotational angle is

$$\theta = 1/\,2[\,\tan{-1}\{\,2a13/(\,a11 - a13)\}] = 0.477658.$$

The second orthogonal matrix R2 is

$$R2 = \begin{bmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix} = = \begin{bmatrix} 0.88807 & 0 & -0.45970 \\ 0 & 1 & 0 \\ 0.45970 & 0 & 0.88807 \end{bmatrix}$$

Then second rotation gives

$$D2 = R_2^{-1}\,D_1\,R_2 = \begin{bmatrix} 5.73205 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2.26795 \end{bmatrix}$$

Thus D2 becomes a diagonal matrix and hence the eigenvalues are 5.73205, −1, 2.26795. The eigenvectors are the columns of R, where

$$R = R_1R_2 = \begin{bmatrix} 0.62796 & -0.70711 & -0.32506 \\ 0.62796 & -0.70711 & -0.32506 \\ 0.45970 & 0 & 0.88807 \end{bmatrix}$$

Hence, the eigenvalues are 5.73205, −1, 2.26795 and the corresponding eigenvectors are (0.62796, 0.62796, 0.45970)T , (−0.70711, −0.70711, 0)T , (−0.32506, −0.32506, 0.88807)T respectively. Note that the eigenvectors are normalized. In this problem, only two rotations are used. This is less than the expected one. The following example shows that at least six rotations are needed to diagonalise a symmetric matrix.

Example 2

Find the eigenvalues and eigenvectors of the symmetric matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 5 & 1 \\ 2 & 1 & 4 \end{bmatrix}$$

using Jacobi's method

Solution:

The largest off-diagonal element is 3 at (1, 2), (2, 1) positions.

The rotational angle θ is given by

$$\tan 2\theta = 2\, a_{12}/(a_{11} - a_{22}) \text{ i.e., } \theta = -0.491397 .$$

Thus the orthogonal matrix R1 is

$$R_1 = \begin{bmatrix} 0.88167 & 0.47186 & 0 \\ -0.47186 & 0.88167 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then the first rotation yields

$$D_1 = R_1^{-1} A R_1 = \begin{bmatrix} 0.60555 & 0 & 1.29149 \\ 0 & 6.60555 & 1.82539 \\ 1.29149 & 1.82539 & 4.00 \end{bmatrix}$$

The largest off-diagonal element of D1 is now 1.82539 situated at (2, 3)and (3,2) positions and hence the rotational angle is

$$\theta = 1/2[\tan^{-1}\{2a_{23}/(a_{22} - a_{23})\}] = 0.47668.$$

The second orthogonal matrix R2 is

$$R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.88908 & -0.45775 \\ 0 & 0.45775 & 0.88908 \end{bmatrix}$$

Then second rotation gives

$$D2 = R_2^{-1} D_1 R_2 = \begin{bmatrix} -0.60555 & 0.59119 & 1.14824 \\ 0.59119 & 7.54538 & 0 \\ 1.14824 & 0 & 3.06017 \end{bmatrix}$$

The largest off-diagonal element in magnitude is 1.14824 which is at the position (1, 3) and (3, 1). The rotational angle θ = 1/ 2[tan−1{ 2a31/(a33 – a11)}]= 0.279829

$$R3 = \begin{bmatrix} 0.96110 & 0 & 0.27619 \\ 0 & 1 & 0 \\ -0.27619 & 0 & 0.96110 \end{bmatrix}$$

$$D3 = \begin{bmatrix} -0.93552 & 0.56819 & 0 \\ 0.56829 & 7.54538 & 0.16328 \\ 0 & 0.16328 & 3.39014 \end{bmatrix}$$

The largest off-diagonal element in magnitude is 0.56829 which is at the position (1, 2) and (2, 1).

The rotational angle θ = 1/ 2[tan−1{ (2 a12/(a11−a22)}] = −0.066600.

$$R4= \begin{bmatrix} 0.99778 & 0.0000 & 0 \\ 0.56829 & 0.99778 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D4 = \begin{bmatrix} -0.93552 & 0.0000 & -0.01087 \\ 0 & 7.58328 & 0.16292 \\ -0.01087 & 0.16292 & 3.39014 \end{bmatrix}$$

The largest off-diagonal element in magnitude is 0.16292 which is at the position (2, 3) and (3, 2). The rotational angle θ = 1/ 2[tan−1{ 2a32/(a33−a22)}] = −0.038776.

$$R5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.99925 & -0.03877 \\ 0 & 0.03877 & 0.99925 \end{bmatrix}$$

$$D5 = \begin{bmatrix} -0.97342 & -0.00042 & -0.01086 \\ -0.00042 & 7.58960 & 0 \\ -0.01086 & 0 & 3.38382 \end{bmatrix}$$

$$R6 = \begin{bmatrix} 1 & 0 & -0.00249 \\ 0 & 1 & 0 \\ 0.00249 & 0 & 1 \end{bmatrix}$$

$$D6 = \begin{bmatrix} -0.97346 & -0.00042 & -0.00001 \\ -0.00042 & 7.58960 & 0 \\ 0.00001 & 0 & 3.38387 \end{bmatrix}$$

This matrix is almost diagonal and hence the eigenvalues are −0.9735, 7.5896, 3.3839 correct up to four decimal places. The eigenvectors are the columns of

$$R = R_1R_2\ R_3R_4\ R_5R_6 = \begin{bmatrix} 0.87115 & 0.47998 & 0.01515 \\ -0.39481 & -0.70711 & -0.54628 \\ -0.27339 & 0.47329 & 0.83747 \end{bmatrix}$$

That is, the eigenvectors corresponding to the eigenvalues −0.9735, 7.5896, 3.3839 are respectively $(0.87115, -0.39418, -0.27339)^T$, $(0.47998, 0.73872, 0.47329)^T$ and $(0.01515, -0.54628, 0.83747)^T$.

Exercise:

1: Eigenvalues and Eigenvectors of a 3x3 Symmetric Matrix

Find the eigenvalues and eigenvectors of the following symmetric matrix AAA using the Jacobi method:

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

2 : Eigenvalues and Eigenvectors of a 3x3 Symmetric Matrix

Find the eigenvalues and eigenvectors of the following symmetric matrix AAA using the Jacobi method:

$$A = \begin{bmatrix} 6 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

3.2. NUMERICAL DIFFERENTIATION

It is the process of calculating the value of the derivative of a function at some assigned value of x from the given set of values (x_i, y_i). To compute dy/dx, we first replace the exact relation $y = f(x)$ by the best interpolating polynomial $y = \phi(x)$ and then differentiate the latter as many times as we desire. The choice of the interpolation formula to be used, will depend on the assigned value of x at which dy/dx is desired.

If the values of x are Equis paced and dy/dx is required near the beginning of the table, we employ Newton's forward formula. If it is required near the end of the table, we use Newton's backward formula. For values near the middle of the table, dy/dx is calculated by means of Stirling's or Bessel's formula. If the values of x are not equispaced, we use Lagrange's formula or Newton's divided difference formula to represent the function.

Hence corresponding to each of the interpolation formulae, we can derive a formula for finding the derivative.

Formulae for Derivatives

Consider the function $y=f(x)$ which is tabulated for the values $(x_i= x_0+ih)$, $i = 0, 1, 2, \ldots n$.

Derivatives using Newton's forward difference formula

Newton's forward interpolation formula is

$$y=y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

Differentiating both sides w.r.t. p, we have

$$\frac{dy}{dp} = \Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \ldots$$

Since $p= \frac{(x - x_0)}{h}$

Therefore $\frac{dy}{dx} = \frac{1}{h}$

Now $$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \cdots\right] \quad (1)$$

At $x = x_0$, p = 0. Hence putting p = 0,

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + \cdots\right] \quad (2)$$

Again differentiating (1) w.r.t x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dp}\right)\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2-36p^2-36p+22}{4!}\Delta^4 y_0 + \cdots\right]\frac{1}{h}$$

Putting p=0, we obtain

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right] \quad (3)$$

Similarly

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right] \quad (4)$$

Derivatives using Newton's backward difference formula

Newton's backward interpolation formula is

$y=y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots$

Differentiating both sides w.r.t. p, we have

$$\frac{dy}{dp} = \nabla y_n + \frac{(2p-1)}{2!}\nabla^2 y_n + \frac{3p^2-6p+2}{3!}\nabla^3 y_n + \ldots$$

Since $p= \frac{(x-x_n)}{h}$

Therefore $\frac{dy}{dx} = \frac{1}{h}$

Now $\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{(2p+1)}{2!}\nabla^2 y_n + \frac{3p^2-6p+2}{3!}\nabla^3 y_n + \cdots\right]$ (1)

At $x = x_n$, p = 0. Hence putting p = 0,

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right] \quad (2)$$

Again differentiating (1) w.r.t x, we get

$\frac{d^2y}{dx^2} = \frac{d}{\text{dp}}\left(\frac{dy}{dp}\right)\frac{dp}{\text{dx}}$

$$= \frac{1}{\text{h}}\left[\frac{2}{2!}\nabla^2 y_n + \frac{6p-6}{3!}\,\nabla^3 y_n\; + \frac{12p^2-36p^2-36p+22}{4!}\nabla^4 y_n + \cdots\right]\frac{1}{\text{h}}$$

Putting p=0, we obtain

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\,\nabla^4 y_n - \cdots\right] \qquad (3)$$

Similarly

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\,\nabla^4 y_n + \cdots\right] \qquad (4)$$

Example 1

Given that

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.4403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{\text{dx}}$ and $\frac{d^2y}{dx^2}$ at x=1.6

Solution

The difference table is:

x	**y**	**Δ**	**Δ^2**	**Δ^3**	**Δ^4**	**Δ^5**	**Δ^6**
1.0	7.989						
		0.414					
1.1	8.403		-0.036				
		0.378		0.006			
1.2	8.781		-0.030		-0.002		
		0.348		0.004		0.001	

1.3	9.129		-0.026		-0.001		0.002
		0.322		0.003		0.003	
1.4	9.451		-0.023		0.001		
		0.299		0.005			
1.5	9.750		-0.018				
		0.281					
1.6	10.031						

The backward difference formula is

$$\frac{dy}{\mathrm{dx}} = \frac{1}{\mathrm{h}}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right]$$

and

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n - \cdots\right]$$

Here h = 0.1, x_n = 1.6, ∇y_n = 0.281, $\nabla^2 y_n$ = − 0.018 etc.

Putting these values in (i) and (ii), we get

$$\frac{dy}{\mathrm{dx}} = \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.012) + \frac{1}{3}(0.005) + \frac{1}{4}(0.002) + \frac{1}{5}(0.003) + \frac{1}{6}(0.002)\right]$$

=2.75

$$\frac{d^2y}{dx^2} = \frac{1}{(0.1)^2}\left[(-0.018) + 0.005 + \frac{11}{12}(0.002) + \frac{5}{6}(0.003) + \frac{137}{180}(0.002)\right]$$

=-0.715

Example 2

Given Table

x	1	2	3	4	5
y	1	4	9	16	25

Solution

The difference table is:

x	y	Δ	Δ^2	Δ^3
1	1			
		3		
2	4		2	
		5		0
3	9		2	
		7		0
4	16		2	
		9		
5	25			

The backward difference formula is

$y=y_n + p\nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \ldots$

Here h = 1, x= 6

$$p = \frac{x - x_n}{h}$$

$$= \frac{6-5}{1} = 1$$

Putting these values, we get

$$y = \left[25 + 9 \ + \frac{2}{2}\ 2\right]$$

$=36$

Example 3

Given Table

x	1	2	3	4	5
y	1	8	27	64	125

Solution

The difference table is:

x	**y**	**Δ**	Δ^2	Δ^3	Δ^4
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		
		61			
5	125				

The backward difference formula is

$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots$

Here h = 1, x= 6

$$p = \frac{x - x_n}{h}$$

$$= \frac{6-5}{1} = 1$$

Putting these values, we get

$y = \left[125 + 61 + \frac{2}{2}\,24 + \frac{6}{6}\,6 + \right]$

$= 216$

Exercise

1. The value of table for *x* and *y*

x	1891	1901	1911	1921	1931	
y	46	66	81	93	101	

2. For the following values of x and y, find the first derivative at x = 4

x	1	2	4	8	10
y	0	1	5	21	27

3. For the following values of x and y, find the first derivative at x = 0.4

x	0.1	0.2	0.3	0.4
y	1.10517	1.22140	1.34986	1.49182

Derivatives using central difference formula

Central difference formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \left(p - \frac{1}{2}\right)\left(\frac{p(p-1)}{3!}\right)\Delta^3 y_{-1} + \ldots$$

Differentiating both sides w.r.t. p, we have

Since $p = \frac{(x - x_0)}{h}$

Therefore $\frac{dy}{dx} = \frac{1}{h}$

Now $\frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{3p^2 - 2p + \frac{1}{2}}{3!}\Delta^3 y_{-1} + \cdots\right]$ (1)

At $x = x_0$, p = 0. Hence putting p = 0,

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{1}{2}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{1}{12}\Delta^3 y_{-1} + \cdots\right] \quad (2)$$

Similarly

$$\frac{d^2 y}{dx^2} = \frac{1}{h^2}\left[\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) - \frac{1}{2}\Delta^3 y_{-1} - \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_0}{2}\right) \ldots\right] \quad (3)$$

Example 1

x	1.0	1.5	2.0	2.5	3.0
y	10.2400	12.3452	15.2312	17.5412	19.3499

i	x	y	Δ	Δ^2	Δ^3
-2	1.0	10.2400			
			2.1052		
-1	1.5	12.3452		0.7808	
			2.8860		-1.3568
0	2.0	15.2312		-0.5760	
			2.3100		0.0747
1	2.5	17.5412		-0.5013	
			1.8087		
2	3.0	19.3499			

Here x=1.55. Let x_0=2.0. Therefore,

$p = \frac{1.55-2.0}{0.5}$= -0.9

$$y_p=\frac{y_0+y_n}{2!}+\left(p-\frac{1}{2}\right)\Delta y_0+\frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}+\left(p-\frac{1}{2}\right)\left(\frac{p(p-1)}{3!}\right)\Delta^3 y_{-1}+\ldots$$

$$=\frac{15.2812+17.5412}{2}+(-0.9)2.31+\frac{(-0.9)(-1.9)}{2!}\frac{(-0.5760-0.5013)}{2}+\frac{1}{6}(-1.4)(-0.9)(-1.9)(0.0747)$$

= 13.5829

Example 2

Use central difference method to find the y from given table

x	20	24	28	32
y	2854	3162	3544	3992

Solution

i	x	y	Δ	Δ^2	Δ^3
-2	20	2854			
			308		
-1	24	3162		74	
			382		-8
0	28	3544		66	
			448		
1	32	3992			

$p = \frac{25-24}{4} = 0.25$

$$y_p = \frac{y_0+y_n}{2!} + \left(p-\frac{1}{2}\right)\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \left(p-\frac{1}{2}\right)\left(\frac{p(p-1)}{3!}\right)\Delta^3 y_{-1} + \ldots$$

$$= \frac{3162+3544}{2} + (-0.25)382 + \frac{(0.25)(-0.75)}{2!}\frac{74+66}{2} + \frac{1}{6}(-0.25)(0.25)(-0.75)(-8)$$

$Y_{0.25} = 3250.875$

Example 3

Use central difference method to find the y for x = 9 from the given table

x	0	4	8	12	16
y	14	24	32	35	40

Solution

x	**y**	**Δ**	Δ^2	Δ^3	Δ^4
0	14				
		10			
4	24		-2		
		8		-3	
8	32		-5		10
		5		7	
12	35		2		
		3			
16	40				

$p = \frac{9-8}{4} = 0.25$

$y_p = \frac{y_0+y_n}{2!} + \left(p - \frac{1}{2}\right)\Delta y_0 + \frac{p(p-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \left(p - \frac{1}{2}\right)\left(\frac{p(p-1)}{3!}\right)\Delta^3 y_{-1} + \ldots$

$= \frac{32+35}{2} + (-0.25)3 + \frac{(0.25)(-0.75)}{2!}\ \frac{-5+2}{2} + \frac{1}{6}(-0.25)(0.25)(-0.75)(7)$

y = 31.21

Exercise

1. Find dy/dx at x = 1 from the following table by constructing a central difference table:

x	0.7	0.8	0.9	1.0	1.1	1.2	1.3
y	0.644218	0.717356	0.783327	0.841471	0.891471	0.932039	0.963558

2. Find dy/dx at x = 0.04 from the following table by constructing a central difference table:

x	0.01	0.02	0.03	0.04	0.05	0.06
y	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

3. Find dy/dx at x = 7.5 from the following table by constructing a central difference table:

x	6	7	8	9
y	1.556	1.690	1.908	2.158

Taylor series:

The Taylor series of a real or complex-valued function $f(x)$, that is infinitely differentiable at a real or complex number a, is the power series

Example 1

Determine the Taylor series at x=0 for $f(x) = e^x$

Solution:

Given: $f(x) = e^x$

Differentiate the given equation,

$f'(x) = e^x$

f''(x) =e^x

f'''(x) = e^x

At x=0, we get

f'(0) = e^0 =1

f''(0) = e^0=1

f'''(0) = e^0 = 1

When Taylor series at x= 0, then the Maclaurin series is

f(x)=f(0)+f'(0)x+f''(0)2!x2+f'''(0)3!x3+….

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots , \qquad -\infty < x < \infty$$

Therefore, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots ,\ -\infty < x < \infty$

Example 2

Evaluate the Taylor Series for f (x) = cos (x) for x = 0.

Solution:

We need to take the derivatives of the cos x and evaluate them at x = 0.

f(x) = cos x ⇒ f(0) = 1

f'(x) = -sin x ⇒ f'(0) = 0

f''(x) = -cos x ⇒ f''(0) = -1

f'''(x) = sin x ⇒ f'''(0) = 0

f''''(x) = cos x ⇒ f''''(4) = 1

f(5)(x) = -sin x ⇒ f(5) (0) = 0

f(6) (x) = -cos x ⇒ f(6)(0) = -1

Therefore, according to the Taylor series expansion;

$$cos\, x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

Example 3

Evaluate the Taylor Series for $f(x) = x^3 - 10x^2 + 6$ at $x = 3$.

Solution:

First, we will find the derivatives of the given function.

$f(x) = x^3 - 10x^2 + 6 \Rightarrow f(3) = -57$

$f'(x) = 3x^2 - 20x \Rightarrow f'(3) = 33$

$f''(x) = 6x - 20 \Rightarrow f''(3) = -2$

$f'''(x) = 6 \Rightarrow f'''(3) = 6$

$f''''(x) = 0$

$= -57-33(x-3)-(x-3)^2+(x-3)^3$

Exercise

1.Write the taylor Series of exponential

2. Use the formula for the coefficients in terms of derivatives to give the Taylor series of $f(z)=e^z$ around z=0.

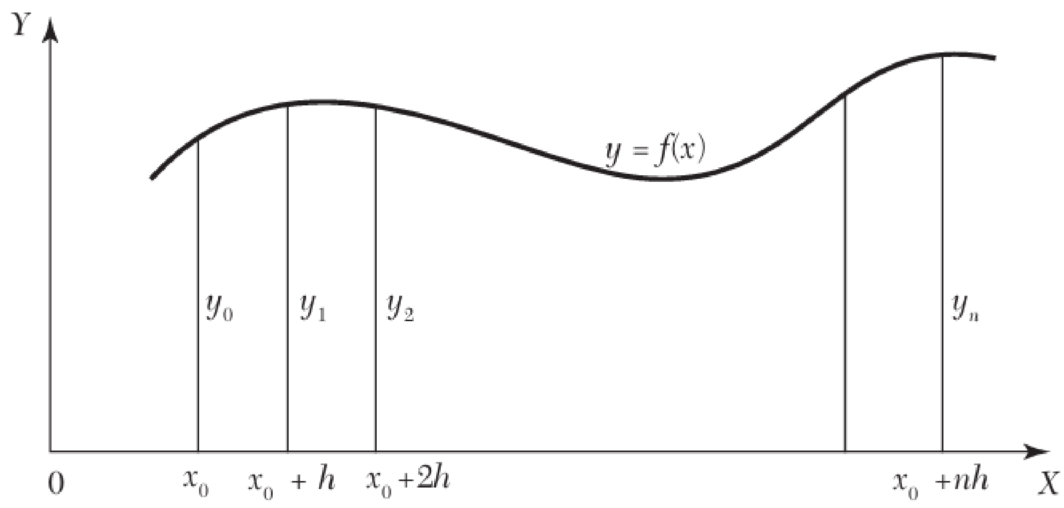
3. Expand $f(z)=z^8e^{3z}$ in a Taylor series around z=0.

3.3. NUMERICAL INTEGRATION

Newton-cotes formula

Let $I = \int_a^b f(x)dx$

Where f(x) takes the values $y_0, y_1, y_2, \ldots y_n$ for $x = x_0, x_1, x_2, \ldots x_n$. Let us divide the interval (a, b) into n sub-intervals of width h so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, … .$x_n = x_0 + nh = b$. Then



$I = \int_{x_0}^{x_0-nh} f(x)dx$ = h$\int_0^n f(x+rh)dr$, Putting x=x_0 +rh, dx=hdr

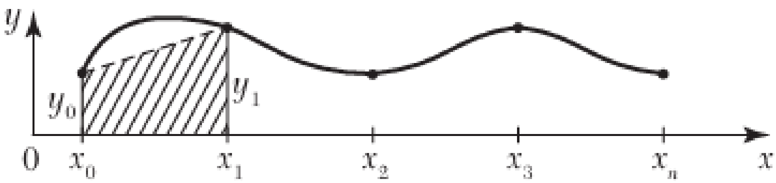
= h$\int_0^n y_0 + r\ \Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \ldots$

Integrating term by term, we obtain

$\int_{x_0}^{x_0-nh} f(x)dx$ = nh $\left[y_0 + \frac{n}{2}\ \Delta y_0 + \frac{n(2n-3)}{12}\ \Delta^2 y_0 + \frac{n(n-2)^2}{12}\ \Delta^3 y_0 \ldots\right]$

This is known as Newton-Cotes quadrature formula. From this gen eral formula, we deduce the following important quadrature rules by taking n = 1, 2, 3, …

I. Trapezoidal rule. Putting n = 1 in (1) and taking the curve through (x_0 , y_0) and (x_1 , y_1) as a straight line i.e., a polynomial of first order so that differences of order higher than first become zero, we get



$\int_{x_0}^{x_0+h} f(x)dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}(y_0 + y_1)$

$\int_{x_0+h}^{x_0+2h} f(x)dx = h\left(y_1 + \frac{1}{2}\Delta y_1\right) = \frac{h}{2}(y_1 + y_2)$

Adding these two integrals, we obtain

$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots. + y_{n-1})$

This is known as the trapezoidal rule.

Example 1

Approximate the area under the curve y = f(x) between x =0 and x=8 using Trapezoidal Rule with n = 4 subintervals. A function f(x) is given in the table of values.

X	0	2	4	6	8
f(x)	3	7	11	9	3

Solution

The Trapezoidal Rule formula for n= 4 subintervals is given as:

$T_4 = (\Delta x/2)[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$

Here the subinterval width $\Delta x = 2$.

Now, substitute the values from the table, to find the approximate value of the area under the curve.

$A \approx T_4 = (2/2)[3 + 2(7) + 2(11) + 2(9) + 3]$

A≈ T_4 = 3 + 14 + 22+ 18+3 = 60

Therefore, the approximate value of area under the curve using Trapezoidal Rule is 60.

Example 2

Approximate the area under the curve y = f(x) between x =-4 and x= 2 using Trapezoidal Rule with n = 6 subintervals. A function f(x) is given in the table of values.

x	-4	-3	-2	-1	0	1	2
f(x)	0	4	5	3	10	11	2

Solution

The Trapezoidal Rule formula for n= 6 subintervals is given as:

$T_6 =(\Delta x/2)[f(x_0)+ 2f(x_1)+ 2f(x_2)+2f(x_3) + 2f(x_4)+2f(x_5)+ f(x_6)]$

Here the subinterval width Δx = 1.

Now, substitute the values from the table, to find the approximate value of the area under the curve.

A≈ $T_6 =(1/2)[0+ 2(4)+ 2(5)+2(3) + 2(10)+2(11) +2]$

A≈ $T_6 =(½) [8 + 10 + 6+ 20 +22 +2] = 68/2 = 34$

Therefore, the approximate value of area under the curve using Trapezoidal Rule is 34.

Example 3

Using trapezoidal method find ∫30ex2dx with step size h = 0.5

Now using trapezoidal method,

T4 $=(\Delta x/2)[f(x_0)+ 2f(x_1)+ 2f(x_2)+2f(x_3) + f(x_4)]$

=0.52[8104.084+2[586.098]]

=0.52[9276.28]

= 2319.07

Error Estimation

$E \leq \frac{(b-a)^2}{12n^2} [\max |f''(x)|]$

Exercise

1. Find out the area under the curve with help of the Trapezoid Rule Formula that passes through the following points

X	0	0.5	1	1.5
Y	5	6	9	11

2. With the help of the Trapezoidal rule, the formula finds the area under the curve y = X sq between x = 0 and x = 4 using the step size of 1

Given, $y = x^2$

h = 1

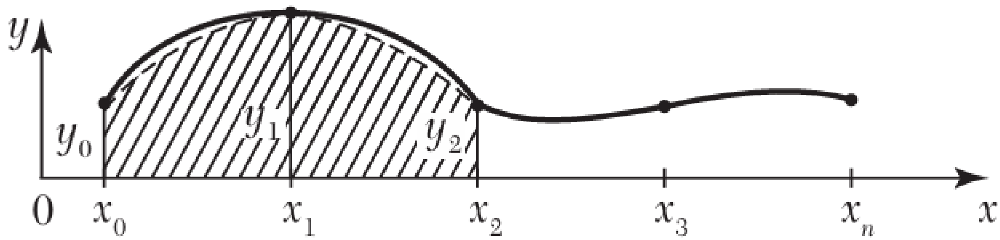
Let's find out the value of y with help of this $y = x^2$

X	0	1	2	3	4
Y	0	1	4	9	16

3. Find the area enclosed by the function f(x) between x = 0 to x = 3 with 3 intervals. f(x) = x

II. **Simpson's one-third rule.**

Putting n = 2 in (1) above and taking the curve through (x_0, y_0), (x_1, y_1), and (x_2, y_2) as a parabola i.e., a polynomial of the second order so that differences of order higher than the second vanish, we get



$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

This is known as the Simpson's one-third rule or simply Simpson's rule and is most commonly used.

Error Estimation

$E \leq \frac{(b-a)^5}{180n^4}$ [max $|f^{(4)}(x)|$]

Example 1

1. Evaluate $\int_0^1 e^x dx$, by Simpson's ⅓ rule.

Let us divide the range [0, 1] into six equal parts by taking h = 1/6.

If $x_0 = 0$ then $y_0 = e^0 = 1$.

If $x_1 = x_0 + h = ⅙$, then $y_1 = e^{1/6} = 1.1813$

If $x_2 = x_0 + 2h = 2/6 = 1/3$ then, $y_2 = e^{1/3} = 1.3956$

If $x_3 = x_0 + 3h = 3/6 = ½$ then $y_3 = e^{1/2} = 1.6487$

If $x_4 = x_0 + 4h = 4/6$ ⅔ then $y_4 = e^{2/3} = 1.9477$

If $x_5 = x_0 + 5h = ⅚$ then $y_5 = e^{5/6} = 2.3009$

If $x_6 = x_0 + 6h = 6/6 = 1$ then $y_6 = e^1 = 2.7182$

We know by Simpson's ⅓ rule;

$\int_a^b f(x)\, dx = h/3\, [(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \ldots + y_{n-2})]$

Therefore,

$\int_0^1 e^x dx$ = (1/18) [(1 + 2.7182) + 4(1.1813 + 1.6487 + 2.3009) + 2(1.39561 + 1.9477)]

= (1/18)[3.7182 + 20.5236 + 6.68662]

= 1.7182 (approx.)

Example 2

Find the solution using Simpson's 1/3 rule

x	0.0	0.1	0.2	0.3	0.4
f(x)	1.0000	0.9975	0.9900	0.9776	0.8604

Solution:

We know that to use Simpson's ⅓ rule, the number of subintervals has to be even.

We have, a = 0.0, b = 0.4, n = 4

First we find that, h=$\frac{b-a}{n}$=0.4−0.04=0.44=0.1

h=$\frac{b-a}{n}$=0.4−0.04=0.44=0.1.

Now putting all these values in the Simpson's ⅓ rule formula, we get

$\int_a^b f(x)dx = h/3[f(x_0)+f(x_n)+4\times(f(x_1)+f(x_3)+\ldots)+2\times(f(x_2)+f(x_4)+\ldots)]$

$=\frac{0.1}{3}\times[f(0.0)+f(0.4)+4\times(f(0.1)+f(0.3))+2\times f(0.2)]$

$\frac{0.1}{3}\times[1+0.8604+4\times(0.9975+0.9776)+2\times0.99]$

$=\frac{0.1}{3}[1+0.8604+4\times(0.9975+0.9776)+2\times0.99]$

$=\frac{0.1}{3}[1+0.8604+7.9004+1.98]=0.13\times[1+0.8604+7.9004+1.98]$

$=\frac{0.1}{3}[11.7408=0.13\times11.7408]$

=0.39136

Example 3

Find the solution using Simpson's 1/3 rule

x	4	4.2	4.4	4.6	4.8	5.0	5.2
f(x)	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Solution:

We have to use Simpson's ⅜ rule, so for that, the number of subintervals must be a multiple of 3.

So we have a=4, b=5.2, n=6

Therefore, h=$\frac{b-a}{n}$=5.2−46=0.2

h=$\frac{b-a}{n}$=5.2−46=0.2

Now putting these values in the Simpson's ⅜ rule formula.

$\int_a^b f$(x)dx=3h/8[f(x_0)+f(x_n)+2×(f(x_3)+f(x_6)+…)+3×(f(x_1)+f(x_2)+f(x_4)+…)]

$\int_a^b f$(x)dx=3h/8[f(x_0)+f(x_n)+2×(f(x_3)+f(x_6)+…)+3×(f(x_1)+f(x_2)+f(x_4)+…)]

=$\frac{3\times0.2}{8}$[f(4)+f(5.2)+2×f(4.6)+3×(f(4.2)+f(4.4)+f(4.8)+f(5.0))]=3×0.28[f(4)+f(5.2)+2×f(4.6)+3×(f(4.2)+f(4.4)+f(4.8)+f(5.0))]

=$\frac{3\times0.2}{8}$[[1.3863+1.6487+(2×1.5261)+3×(1.4351+1.4816+1.5686+1.6094)]=0.68[1.3863+1.6487+(2×1.5261)+3×(1.4351+1.4816+1.5686+1.6094)]

=$\frac{0.6}{8}$[1.3863+1.6487+3.0522+18.2841]=0.68[1.3863+1.6487+3.0522+18.2841]

=$\frac{0.6}{8}$×24.3713=0.68×24.3713

=1.8278475

Exercise

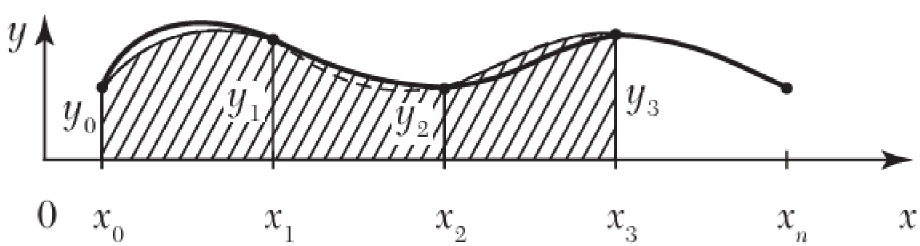
1. Evaluate the integral $\int_0^4 (x^2+1)\,dx$ using Simpson's 1/3 rule with 4 subintervals.

2. Approximate $\int_1^2 \frac{1}{x}dx$ using Simpson's 1/3 rule with 6 subintervals.

3. Use Simpson's 1/3 rule to find the value of $\int_0^\pi sin(x)\,dx$ taking 4 subintervals.

III. **Simpson's 3 / 8 Rule**

The Simpson's 3 / 8 rule is another method that can be used for numerical integration. This numerical method is entirely based on the cubic interpolation instead of the quadratic interpolation. This rule can be represented by the formula that is mentioned below.



$\int_a^b f(x)\,dx = 3h/8\,[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \ldots + y_{n-3})]$

Example 1

1. Find Solution using Simpson's 3/8 rule

x	1.4	1.6	1.8	2	2.2
y	4.0552	4.953	6.0436	7.3891	9.025

Solution,

$\int y dx = \frac{3h}{8}[(y_0+y_4)+2(y_3)+3(y_1+y_2)]$

$\int y dx = \frac{3\times 0.2}{8}[(4.0552+9.025)+2\times(7.3891)+3\times(4.953+6.0436)]$

$\int y dx = \frac{3\times 0.2}{8}[(4.0552+9.025)+2\times(7.3891)+3\times(10.9966)]$

$\int y dx = 4.5636$

Solution by Simpson's 3/8 Rule is 4.5636

Example 2

Find Solution using Simpson's 3/8 rule

x	0	0.1	0.2	0.3	0.4
y	1	0.9975	0.99	0.9776	0.8604

Solution,

$\int y dx = \frac{3h}{8}[(y_0+y_4)+2(y_3)+3(y_1+y_2)]$

$\int y dx = \frac{3\times 0.1}{8}[(1+0.8604)+2\times(0.9776)+3\times(0.9975+0.99)]$

$\int y dx = \frac{3\times 0.1}{8}[(1+0.8604)+2\times(0.9776)+3\times(1.9875)]$

$\int y dx = 0.36668$

Solution by Simpson's 3/8 Rule is 0.36668

Example 3

Find Solution of an equation 1/x using Simpson's 3/8 rule x1 = 1 and x2 = 2 Step value (h) = 0.25

x	1	1.25	1.5	1.75	2
y	1	0.8	0.6667	0.5714	0.5

Solution,

$\int y dx = \frac{3h}{8}[(y_0+y_4)+2(y_3)+3(y_1+y_2)]$

$\int y dx = \frac{3\times0.25}{8}[(1+0.5)+2\times(0.5714)+3\times(0.8+0.6667)]$

$\int y dx = \frac{3\times0.25}{8}[(1+0.5)+2\times(0.5714)+3\times(1.4667)]$

$\int y dx = 0.6603$

Solution by Simpson's 38 Rule is 0.6603

Exercise

1. Find Solution of an equation 2x^3-4x+1 using Simpson's 3/8 rule x1 = 2 and x2 = 4 Step value (h) = 0.5

x	2	2.5	3	3.5	4
y	9	22.25	43	72.75	113

2. Approximate $\int_1^2 \frac{1}{x}$dx using Simpson's 3/8 rule with 6 subintervals.

3. Use Simpson's 3/8 rule to find the value of $\int_0^\pi sin(x)\,dx$ taking 6 subintervals.

UNIT – IV

4.1. Ordinary Differential Equation:

Suppose we want to find the numerical solution of the equation

$$\frac{dy}{dx} = f(x, y) \dots\dots\dots (1)$$

With $y(x_0) = y_0$

y(x) can be expanded about the point x_0 in taylor series as

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!}\,[y'(x)]_{x_0} + \frac{(x - x_0)^2}{2!}\,[y'(x)]_{x_0} + \cdots \dots \dots.$$

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!} y_0{}' + \frac{(x - x_0)^2}{2!} y_0{}'' + \cdots \dots \dots.$$

Putting $x = x_1 = x_0 + h$ we have

$$y(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' +$$

In this y_0', y_0'', y_0''' can be found by differentiating equation (1)

Thus

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots . (2)$$

Once y_1 has been calculated

y_1', y_1'', y_2''' can be found from equation (2)

Similarly expanding y(x) about the point x_1 we get

$$y_3 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots . (3)$$

In general expanding y(x) at a point x_m we get

$$y_{m+1} = y_m + \frac{h}{1!} y'_m + \frac{h^2}{2!} y''_m + \frac{h^3}{3!} y'''_m + \cdots . (4)$$

Problems

1. Solve $\frac{dy}{dx} = x + y$ given y(1) = 0 and get y(1.1), y(1.2) by taylor series method. Compare your result with the explicit solution.

Given: $x_0 = 1, y_0$=0 h=0.1

$y' = x + y$

$y'' = 1 + y'$

$y''' = y''$

$y^{iv} = y'''$

$y_0 = 0$

$y'_o = x_0 + y_o = 1 + 0 = 1$

$y''_o = 1 + y'_o = 1 + 1 = 2$

$y'''_o = y''_o = 2$

$y^{iv}_o = 2$

By Taylor Series

$y_1 = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \frac{h^4}{4!} y^{iv}_0$

$y(1.1) = 0.1 + 0.01 + 0.00033$+0.00000833+0.000000166

$y(1.1) = 0.11033847$

Now take $x_0 = 1.1\ \ h = 0.1$

$$y_2 = y_1 + \frac{h}{1!} y'_1 + \frac{h^2}{2!} y''_1 + \frac{h^3}{3!} y'''_1 + \frac{h^4}{4!} y^{iv}_0$$

$$y_1 = 0.11033847$$

$$y_1' = x_1 + y_1 = 1.21033847$$

$$y_1'' = 1 + y_1' = 2.21033847$$

$$y_1''' = y_1'' = y_o^{iv} = y_o^{v} = 2.21033847$$

$y_2 = y(1.2) = 0.11033847 + \frac{0.1}{1}(1.21033847) + \frac{(0.1)^2}{2}(2.21033847) + \frac{(0.1)^3}{2}(2.21033847) + \frac{(0.1)^4}{2}(2.21033847)$

=0.11033847+0.121033847+2.21033847(0.005+0.00166666+0.00000008333)

$$y(1.2) = 0.24280160$$

Exact solution

$$\frac{dy}{dx} = x + y$$

$$y = -x - 1 + 2e^{x-1}$$

$$y(1.1) = -1.1 - 1 + 2e^{0.1} = 0.11034$$

$$y(1.2) = -1.2 - 1 + 2e^{0.2} = 0.2428$$

X	Taylor result	Exact result
1.1	0.11033847	0.11034
1.2	0.24280160	0.2428

2. Using Taylor series method find corrected to 4 decimal places the value of $y(0.1)$ given $\frac{dy}{dx} = x^2 + y^2$ and y(0)=1, h=0.

Given $y(x_o) = y_0 \quad y(0) = 1 = 0.$

$x_0 = 0 \;\; y_0 = 1 \; h = 0.1$

$x_1 = 0.1 \;\; y_1 = y(0.1) = ??$

$$y' = x^2 + y^2$$

$y'' = 2x + 2yy'$

$y''' = 2 + 2yy' + 2(y')^2$

$y''' = 2 + 2yy'' + 2(y')^2$

$y^{iv} = 2yy''' + 2y'y'' + 4y'y'' = 2yy'' + 6y'y''$

$y_0' = x_0^2 + y_0^2 = 0 + 1 = 1$

$y_0''=2x_0 + 2y_0y_0'$=0+2=2

$y_0'''=2 + 2y_0y_0'' + 2(y_0')^2$= 2+4+2 = 8

$y_0^{iv}=2y_0y_0'' + 6y_0'y_0''$=16+12 =28

By Taylor's series method

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv}$$

$y(0.1) = 1 + 0.1 + 0.01 + 0.00133333 + 0.000116666 = 1.11144999$

3. using Taylor series method, find y at x=0.1,0.2 correct to three significant details given.

$\frac{dy}{dx} - 2y = 3e^x$ y(0) =0

Here $x_0 = 0, y_0 = 0, x_1 = 0.1, x_2 = 0.2, x_1 = 0.1$

$y' = 2y + 3e^x$

$y'' = 2y' + 3e^x$

$y''' = 2y'' + 3e^x$

$y^{iv} = 2y''' + 3e^x$

$y_0' = 2y_0 + 3e^{x_0} = 3$

$y_0'' = 2y_0' + 3e^{x_0}{}' = 9$

$y_o''' = 18 + 3 = 21$

$y_o^{iv} = 42 + 3 = 45$

$$y_1 = y_0 + \frac{h}{1}y_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4}{24}y_0^{iv}$$

$$y(0.1) = \ y_1 = 0 + (0.1)(3) + \frac{0.01}{2}(9) + \frac{0.001}{6}(21) + \frac{0.0001}{24}(45) +$$

=0.3+0.045+0.0035+0.0001875+

=0.3486875=0.349

$y_1' = 2y_1 + 3e^{x_1} = 0.3486875X2 + 3e^{0.1} = 4.012887$

$y_1'' = 2y_1' + 3e^{x_1} = 0.3486875X2 + 3e^{x_1} = 11.025744$

$y_1''' = 2y_1'' + 3e^{x_1} = 25.3670608$

$$y_2 \ = y(0.2) = \ y_1 + \frac{h}{1}y_1' + \frac{h^2}{2}y_1'' + \cdots.$$

$$= 0.3486875 + (0.1)(4.012887) + \frac{0.01}{2}(11.341286) + \frac{0.001}{6}(25.99808) + \cdots.$$

=0.8110156=0.811

Exercises:

1. Using Taylor method, compute y(0.2) y(0.4) correct to 4 decimal places given $\frac{dy}{dx} = 1 - 2xy, y(0) = 0$

   Solution: y(0.2)=0.194752003, y(0.4)=0.35988

2. Using Taylor series method find y(1.1) and y(1.2) correct to four decimal places given $\frac{dy}{dx} = xy^{1/3}$

   Solution: $y(1.1) = 1.10681, y(1.2) = 1.22772$

3. Using Taylor series method find y(1.1) and y(1.2) correct to four decimal places given $\frac{dy}{dx} - 2y = 3e^x$

Solution: $y(0.1) = 0.349, y(0.2) = 0.811$

4.2. Taylor Series Method for Higher Order Differential Equations:

$$\frac{d^2y}{dx^2} = \left(f(x, y, \frac{dy}{dx}\right) \dots (1)$$

$$y'' = f(x, y, y')$$

With $y(x_o) = y_0 \dots \dots .. (i)$

And $y'^{(x_o)} = y_0' \;\; \dots . (ii)$

Put $y' = P \dots (2)$

$y'' = P'$ so equation 1 becomes

$$P' = f(x, y, P) \dots \dots \dots . (3)$$

Initial conditions (i) and (ii) become

$$y(x_0) = y_0 \dots \dots .. (iii)$$

Hence we need to solve first order differential equations subject to (iii) & (iv) conditions

Taylor algorithm for equation (3)

$$P_1 = P_0 + hP_0' + \frac{h^2}{2!}P_0'' + \frac{h^3}{3!}P_0''' +$$

Where h is

$x = x_1 {=} x_0 + h$

Taylor algorithm for equ (2) is

$P_1 = P_0 + hP_0' + \frac{h^2}{2!}P_0'' +$

Where h is

$x = x_1 = x_0 + h$

Taylor algorithm for equ (2) is

$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' +$

$$y_1 = y_0 + hP_0 + \frac{h^2}{2!} P_0' + \frac{h^3}{3!} P_0''' \ldots \ldots (5)$$

Differentiating equation (3) successively we get P'', P''' etc. so the values of, P_0', P_0'', P_0''' can be calculated.

$$P_2 = P_1 + hP_1' + \frac{h^2}{2!} P_1'' + \frac{h^3}{3!} P_1''' \ldots \ldots (6)$$

$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' +$

$$y_2 = y_1 + hP_1 + \frac{h^2}{2!} P_1' + \frac{h^3}{3!} P_1'' + \cdots (7)$$

Problem 1: Evaluate the values of y(0.1) and y(0.2) given $y'' - x(y')^2 + y^2 = 0$, y(0)=1, y'(0)=0 by Taylor series method

Solution:

$$y'' - x(y')^2 + y^2 = 0 \ldots \ldots (1)$$

Put y' = Z

Hence the equation reduces to

Z'-x$z^2 + y^2 = 0$

Z'=x$z^2 - y^2 \ldots \ldots \ldots \ldots \ldots (2)$

By Initial condition

$y_0 = y(0) = 1$

$z_0 = y_0' = 0$

Now $z_0 = z(0) = 0$ and $x_0 = 0$

$z_1 = z_0 + hz_0' + \frac{h^2}{2!} yz_0'' +$

From equ(2) we get

$Z' = xz^2 - y^2$

$Z'' = z^2 + 2xzz' - 2yy'$

$z_0' = x_0 z^2 - y_0^2 = -1$

$z_0'' = z_0^2 + 2x_0 z_0{}^2 z_0' - 2y_o y_0' = 0$

$z_0'' = 2$

$z_1 = -0.0997$

By Taylor series for y1,

$y_1 = y(0.1) = y_0 + hy_0' + \frac{h^2}{2} y_0'' + \cdots$

$= 1 + (0.1)(z_0) + \frac{(0.01)}{2} z_0' + \frac{0.001}{6}(0) + \cdots$

=1-0.005=0.995

Similarly,

$y_2 = y(x_2) = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \cdots$

$= 0.995 + \frac{(0.1)}{2} z_1 + \frac{(0.01)}{2} z_1' + \frac{(0.001)}{6} z_1'' +$

$z_1' = x_1 z_1^2 - y_1^2$

$= (0.1)(-0.0997)(-0.0997) - (0.995)^3$=-0.9890

$$z_1' = -0.1087$$

$$y_2 = 0.995 + \frac{(0.1)}{1}(-0.0997) + \frac{(0.01)}{2}(0.9890) + \frac{(0.001)}{6}(-0.1687) + \cdots . = 0.9801$$

$$y(0.1) = 0.9950$$

$$y(0.2) = 0.9801$$

Exercise

Solve y''=$y + xy'$ given y(0)=1, $y_0' = 0$ and calculate $y(0.1)$

Solution: $y(0.1) = 1.00501252$

4.3. Euler's Method

In solving a first order differential equation by numerical methods, we come across two types of solution

(i) A series solution of y in terms of x which wil yield the value of y for a particular value of x by direct substitution in the series solution.

(ii) Values of y at specified valued x

So the methods such as Euler, Runge kutta comes under the second category.

The methods of second category are called step by step methods because the values of y are calculated by short steps ahead of equal interval h of the independent variable x.

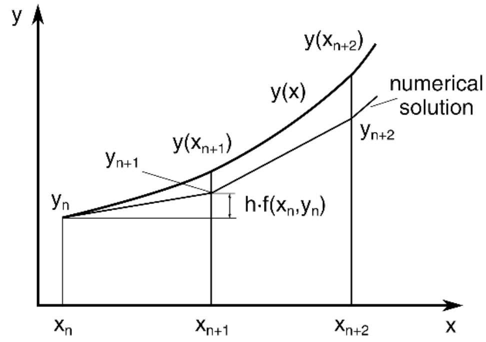
AIM: To solve $\frac{dy}{dx} = F(x, y)$ with the initial condition

$$y(x_0) = y_0 \ldots..(1)$$

Let us take the points

$$x = x_0, x_1, x_2 \ldots.$$

$$i.e., x_i = x_0 + ih$$



Let the solution of the differential solution be denoted by the continuous graph $P_0(x_0, y_0)$ lies on the curve. We require the value of y of the curve at $x = x_1$

The equation of the tangent at (x_0, y_0) to the curve is

$y - y_0 = y'_{(x_0,y_0)}\ (x - x_0)$

$y - y_0 = f(x_0, y_0)\ (x - x_0)$

$y = y_0 + f(x_0, y_0)(x - x_0) \ldots (2)$

This y is the value of y on the tangent corresponding to x=x. in the interval (x_0, x_1) the curve is approximated by the tangent.

$y_1 = y_0 + f(x_0, y_0)(x - x_0)$

$y_1 = y_0 + hy'_o \quad h = x_1 - x_0$

Again we approximate curve by the line (x_1, y_1) and whose slope is $f(x_1, y_1)$

$y_2 = y_1 + hf(x_1, y_1) = y_1 + hy'_1$

Thus

$$y_{n+1} = y_n + hf(x_n, y_n) \quad n = 0,1,2 \ldots ..$$

This formula is called Euler's algorithm

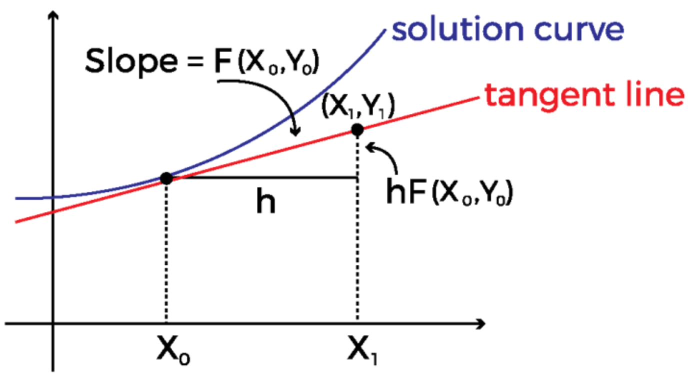
In otherwords

$$y(x + h) = y(x) + hf(x, y)$$

In this method the actual curve is approximated by a sequence of short straight lines. As the interval increases the straight ling deviates much from the actual curve. Hence the accuracy cannot be obtained as the number of interval's increase.

4.4. IMPROVED EULER'S METHOD

Let the tangent at (x_0, y_0) to the curve be P_oA. In the interval (x_1, y_1) by previos euler's method use approximate the curve by the tangent P_oA



$y_1^1 = h_0 + hf(x_0, y_0)$

$y_1^1 = m_1Q_1$

$Q_1\left(x_1, y_1^{(1)}\right)$ let Q_1C be the line at Q_1 whose slope is $f\left(x_1, y_1^{(1)}\right)$. Now take the average of the slopes at P_o and Q_1 i.e.,

$$\frac{1}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

Now draw a line P_oD through $P_o(x_0, y_0)$ with this as the slope

$$y - y_0 = \frac{1}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right](x - x_0) \ldots (2)$$

This line intersects $x = x_1$at

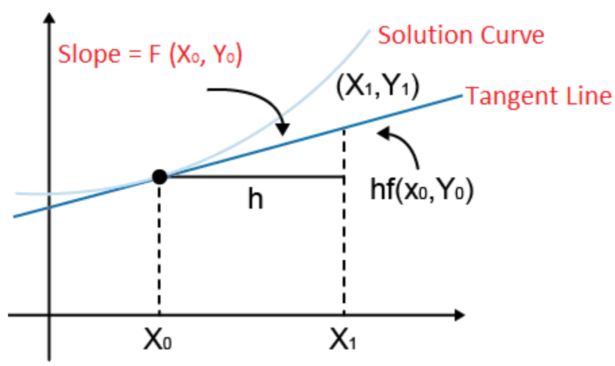
$$y_1 = y_0 + \frac{1}{2}h\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

In general

$$y_{n+1} = y_n + \frac{1}{2}h[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n)]$$

This is improved Euler's method

4.5. MODIFIED EULER METHOD



In the improved Euler method we averaged the slopes whereas in modified euler method we will average the points

Let $P_o(x_0, y_0)$ be the point on the solution curve.

Let P_oA be the tangent to the curve now this pointmeet the ordinate at

$$x = x_0 + \frac{1}{2}hf(x_0, y_0) \dots .(1)$$

Now the slope at N_1 is

$$f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hf(x_0, y_0)$$

Now draw the line $P(x_0, y_0)$ with this slope as the slope. Let this line meet $x = x_1$ at $k_1 = \left(x_1, y_1^{(1)}\right)$. This $y_1^{(1)}$ is taken as approximate value of y at $x = x_1$

$$y_1^{(1)} = y_0 + h\left[f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hf(x_0, y_0)\right]$$

In general

$$y_{n+1}^{(1)} = y_n + h\left[f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right] \ldots (2)$$

$$y(x+h) = y(x) + h\left[f(x + \frac{1}{2}h, y + \frac{1}{2}f(x, y)\right]$$

This is modified Euler's formula.

Problem 1

Given $y' = -y$ and $y(0) = 1$ determine the values of y at x=(0.01) to (0.04) by Euler method

Solution: $y' = -y$

$y(0) = 1$

$f(x, y) = -y$

$x_0 = 1, y_0 = 1, x_1 = 0.011, x_2 = 0.02, x_3 = 0.03, x_4 = 0.04, h = 0.01$

$y_{n+1} = y_n + hf(x_n, y_n)$

$y_1 = y_0 + hf(x_0, y_0) = 1 - 0.01 =$0.99

$y_2 = y_1 + hf(x_1, y_1) = 0.9801$

$y_3 = y_2 + hf(x_2, y_2) = 0.9703$

$$y_4 = y_3 + hf(x_3, y_3) = 0.9606$$

X	Y
0	1
0.01	0.9900
0.02	0.9801
0.03	0.9703
0.04	0.9606

Exercises:

1.Using Euler's Method, solve $y' = x + y, y(0) = 1 \; for \; x = 0.0(0.2)(1.0)$

Solution:

X	Y
0.2	1.48
0.4	1.856
0.6	2.3472
0.8	2.97664
1.0	3.771968

2. Solve numerically $y' = y + e^x, y(0) = 0 \; for \; x = 0.2{,}0.4$ by improved Euler method

Solution:

$y_1 = 0.24214, y_2 = 0.59116$

3. Given $y' = x^2 - y, y(0) = 1$ find correct to four decimal places the values of y(0.1) by improved Euler method.

Solution: y(0.1)=0.9055

4.compute y at x=0.25 by modified euler method given $y' = 2xy, y(0) = 1$

Solution: y(0.25) = 1.0645

5. Using modified euler method find $y(0.2), y(0.1)$

Solution: $y(0.2) = 1.25026, y(0.1) = 1.1105$

6. Solve the equation $\frac{dy}{dx} = 1 - y$ given $y(0) = 0$ using modified Euler's method and tabulate solutions at x=0.1,0.2 and 0.3. compare your results with the exact solution. Also, get the solutions by Improved Euler method

X	Modified Euler	Improved Euler	Exact Solution
0.1	0.095	0.095	0.09516
0.2	0.18098	0.18098	0.18127
0.3	0.258787	0.258787	0.25918

4.6.RUNGE KUTTA METHOD

In Runge kutta method the derivatives of higher order are not required and we require only the given function values at different points.

Second order Runge kutta method

To solve $\frac{dy}{dx} = f(x, y)$

Given: $y(x_0) = y_0$…..(1)

Proof:

By Taylor series we have

$$y(x + h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \cdots..(2)$$

Differentiating equation (1)wrt x

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = fx + y'fy = f_x + ff_y \dots\dots(3)$$

Using the values of $y' and\ y'' got\ from\ equ\ 1\ and\ 3\ in\ 2\ we\ get$

$$y(x+h) - y(x) = hf + \frac{1}{2}h^2\left[f_x + ff_y\right] + 0(h^3)$$

$$\Delta y = hf + \frac{1}{2}h^2\left[f_x + ff_y\right] + 0(h^3) \dots.(4)$$

Let $\Delta_1 y_1 = k_1 = f(x,y).\Delta x = hf(x,y) \dots..(5)$

$\Delta_2 y_1 = k_2 = hf(x+mh, y+mk_1) \dots..(6)$

And let $\Delta y = ak_1 + bk_2$………(7)

Where a,b and m are constants to be determined

Expanding $k_2\ and\ \Delta y$ in powers of h expanding k_2 by Taylor series for two variables we have

$k_2 = hf(x+mh, y+mk_1)$

$$= h[f(x,y) + \left(mh\frac{\partial}{\partial x} + mk_1\frac{\partial}{\partial y}\right)f + \frac{\left(mh\frac{\partial}{\partial x} + mk_1\frac{\partial}{\partial y}\right)^2 f}{2!} + \cdots \dots.$$

$$h[f + mhf_x + mhff_y + \frac{\left(mh\frac{\partial}{\partial x} + mk_1\frac{\partial}{\partial y}\right)^2 f}{2!} + \cdots$$

since $k_1 = hf$

=hf+$mh^2\left(f_x + ff_y\right) + \cdots$.. higher powers of h ………………..(9)

Sub k_1, k_2 in 7

$$\Delta y = ahf + b(hf + mh^2\left(f_x + ff_y\right) + 0(h^3)$$

$$\Delta y = (a+h)bf + bmh^2\left(f_x + ff_y\right) + 0(h^3) \dots\dots(10)$$

Equating Δy from 4 and 10 we get

a+b=1 and $b_m = \frac{1}{2}$

from a+b=1

a = 1-b

also m=$\frac{1}{2b}$

where $k_1 = hf(x,y)$

$$k_2 = hf\left(x + \frac{h}{2b}, y + \frac{hf}{2b}\right)$$

Δy=y(x+h)-y(x)

$$y(\mathrm{x}+\mathrm{h}) = \mathrm{y(x)} + (1-\mathrm{b})\mathrm{hf} + \mathrm{bhf}\left(x + \frac{h}{2b}, y + \frac{hf}{2b'}\right)$$

$$y_{n+1} = y_n + (1-b)hf(x_n, y_n) + \mathrm{bhf}\left(x_n + \frac{h}{2b}, y_n + \frac{hf}{2b'}\right) + 0(h^3)$$

From this general second order Runge kutta formula, setting a=0,b=1 we get second order Runge Kutta algorithms

$$k_1 = nf(x,y)$$

$$k_2 = nf\left(x + \frac{1}{2}h, y + \frac{1}{2}k_1\right)$$

$$k_3 = nf\left(x + \frac{1}{2}h, y + \frac{1}{2}k_2\right)$$

$$k_4 = nf(x + h, y + k_3)$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y(x+h) = y(x) + \Delta y$$

Exercises

Obtain the values of y at x=0.1,0.2 using runge kutta method (i) second order (ii) fourth order for the differential equation

$y' = -y$ given y(o)=1

Solution

F(x,y) = -y

$$x_0 = 0, y_0 = 1, x_1 = 0.1, x_2 = 0.2$$

Second order

$$k_1 = hf(x_0, y_0) = (0.1)(-y_0) = (0.1)(-1) = -0.1$$

$k_2 = nf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$

$= (0.1)f(0.05{,}0.95) = (0.1)(-0.95)$

$\Delta y = -0.095$

$y_1 = y_0 + \Delta y = 1 - 0.095 = 0.905$

Again, starting from (0.1,0.905) replacing (x_0, y_0) by (x_1, y_1) we get

$k_1 = (0.1)f(0.1{,}0.905) = -0.0905$

$k_2 = nf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right)$

$k_2 = (0.1)(-0.85975)$=-0.085975= Δy

$y_2 = y(0.2) = y_1 + \Delta y = 0.905 - 0.085975 = 0.819025$

Fourth Order

$k_1 = hf(x_0, y_0) = (0.1)(-1) = -0.1$

$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$

$(0.1)f(0.05{,}0.95) = (0.1)(-0.95) = -0.095$

$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$

$0.1f(0.05,0.9525)$

$= (0.1)(-0.9525) = -0.09525$

$k_4 = nf(x_0 + h, y_0 + k_3\ )$

$= (0.1)f(0.1,0.90475) = -0.090475$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(-0.1 + 2(-0.095) + 2(-0.09525) + (-0.090475) = -0.0951625$$

$y_1 = y_0 + \Delta y = 1 - 0.951625 = 0.9048375$

Replacing (x_0, y_0) by (x_1, y_1)

$k_1 = hf(x_1, y_1) = (0.1)f(0.1,0.9048375) = -0.09048375$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$= (0.1)f(0.15,0.8595956) = (0.1)(-0.8595956)$

$k_2 = -0.08595956$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right)$$

$k_3 = 0.1f(0.15,0.8618577)\sim = -0.08618577$

$$k_4 = \frac{1}{2}k_2\left(x_1 + h, y_1 + \frac{1}{2}k_2\right)$$

$= (0.1)(0.2,03.8186517) = (0.1)(-0.8186517)$

$k_4 = -0.08186517$

$$\Delta y = \frac{1}{6}(-0.09048375 - 2 \times 0.08595956) - 2(0.08618577) - 0.08186517$$

$= -0.086106596$

$y_2 = y(0.2) = y_1 + \Delta y = 0.9048375 - 0.086106596 = 0.81873090$

X	Second Order	Fourth order
0.1	0.905	0.9048375
0.2	0.819025	0.81873090

2.Compute y(0.3) given $\frac{dy}{dx} + y + xy^2 = 0, y(0) = 1$ by taking h=0.1 using Runge kutta method of fourth order.

Solution:

$y' = -(xy^2 + y) = f(x,y)$

$$x_0 = 0, y_0 = 1, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, y_3 =??$$

For 1st Interval

$k_1 = hf(x_0, y_0) = (0.1)[-(x_0 y_0^2 + y_0)] = -0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right) = (0.1)f(0.05, 0.95) = -0.0995$

$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1)f(0.05, 0.95025) = -0.0995$

$k_4 = hf(x_0 + h, y_0 + k_3) = -0.0982$

$y_1 = 1 + \frac{1}{6}[-0.1 + 2(-0.0995) + 2(-0.0995) - 0.0982] = 0.9006$

Now $hf(x_1, y_1) = (0.1)f(0.1, 0.9006) = -0.0982$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_1\right)$

$= (0.1) - f(0.15, 0.8526) = -0.0960$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_2\right) = (0.1)f(0.15, 0.8526) = -0.0962$

$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)f(0.2, 0.8044) = -0.0934$

$$y_2 = y_1 + \frac{1}{6}[-0.0982 + 2(-0.960) + 2(-0.0962) + (-0.0934)] = 0.8046$$

$(x_2, y_2) = (0.2, 0.8046)$

$k_1 = hf(0.2, 0.8046) = -0.0934$

$k_2 = hf(0.25, 0.7579) = -0.0902$

$k_3 = hf(0.25, 0.7595) = -0.0904$

$k_4 = hf(0.3, 0.7142) = -0.0867$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y(0.3) = 0.7144$

3.Using Runge kutta method of fourth order, find y(0.8) correct to 4 decimal places if $y' = y - x^2$, y(0.6)=1.7379

Solution

$x_0 = 0.6, y_0 = 1.7379, h = 0.1$

$x_1 = 0.7, x_2 = 0.8, f(x, y) = y - x^2$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_0, y_0) = (0.1)f(0.6, 1.7379) = 0.1378$

$k_2 = hf(0.65, 1.8068) = 0.1384$

$k_3 = (0.1)f(0.65, 1.8071) = 0.1385$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 + f(0.7, 1.8764) = 0.1386$

$y(0.7) = y_1 + \frac{1}{6}[0.1378 + 2(0.1384) + 2(0.1385) + 0.1386] = 1.8763$

$y_2 = y(0.8)$

$(x_1, y_1) = (0.7, 1.8763)$

$k_1 = (0.1)[1.8763 - (0.7^2)] = 0.1386$

$k_2 = hf(0.75, 1.9456) = 0.1383$

$k_3 = (0.1)f(0.75, 1.9455) = 0.1383$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1 + f(0.8, 2.0146) = 0.1375$

$y_2 = 1.8763 + \frac{1}{6}[0.1386 + 2(1.1383) + 2(1.1383) + 0.1375] = 2.0145$

$y_2 = y(0.8) = 2.0145$

EXERCISES

Using Runge kutta method of fourth order find the values of y. when x=0.2 given $y' = x + y$

Solution:

$x_0 = 0, y_0 = 1, h = 0.1, k_1 = 0.1, k_2 = 0.11, k_3 = 0.1105, k_4 = 0.12105,\ y_1 = 1.1103,\ \ y_1 = 1.1103$

At $(x_1, y_1) k_1 = 0.12103, k_2 = 0.132082, k_3 = 0.132634, k_4 = 0.144293,\ \ y_2 = 1.2428$

4.7. Solution of second order differential equations by Runge-Kutta method

Let us consider the second order differential equation of the form

$$\frac{d^2y}{d^2x} = g\left(x, y, \frac{dy}{dx}\right) .. (1)$$

If we put $\frac{dy}{dx} = z$ then $\frac{d^2y}{dx^2} = \frac{dz}{dx}$ ... (2)& (3)

Sub 2 and 3 in 1 we get

$\frac{dz}{dx} = g(x, y, z)$ and $\frac{dy}{dx} = z, \frac{dy}{dx} = z = f(x, y, z) .. (4)$

$$\frac{dz}{dx} = g(x, y, z)$$

Equ 4 and 5 gives a system of simultaneous differential equations which can be solved a sbelow. So instead of solving the second order differential equation given by 1 it is enough if we solve the system of equ 4 and 5. To solve this system of differential equations at an interval of h, the increments in y and z for the first increment in x computed by using the following formulae.

$$k_1 = hf(x_{0,} y_{0,} z_0);\ l_1 = hg(x_{0,} y_{0,} z_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right): l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right): l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3): l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$$

Now $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ and $\Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

In a similar manner we can find the next increment by replacing $x_{0,} y_{0,} z_0$ by $x_{1,} y_{1,} z_1$ and so on.

Examples

Solve $\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} - x\left(\frac{dy}{dx}\right)^2 + y^2 = 0$ using Runge kutta method for x=0.2 correct to 4 decimal places. Initial conditions are x=0,y=1,$y' = 0$

$$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} - x\left(\frac{dy}{dx}\right)^2 + y^2 = 0$$

If we put $\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{z}$ then $\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} = \frac{\mathrm{dz}}{\mathrm{dx}}$

$$\frac{\mathrm{dz}}{\mathrm{dx}} = xz^2 - y^2$$

Let $\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{z} = f(x, y, z); \frac{\mathrm{dz}}{\mathrm{dx}} = xz^2 - y^2 = g(x, y, z)$

Also we are given that x=0,y=1,$y' = 0$ (or) $z_0 = 0, h = 0.2$

Now

$k_1 = hf(x_{0,}y_{0,}z_0) = hz_0 = 0$

$l_1 = hg(x_{0,}y_{0,}z_0) = h(x_0z_0^2 - y_0^2) = (0.2)(0-1) = -0.2$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = h\left(z_0 + \frac{l_1}{2}\right) = -0.02$

$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = h\left[\left(x_0 + \frac{h}{2}\right)\left(z_0 + \frac{l_1}{2}\right) - y_0 + \frac{k_1}{2}\right]$

$= (0.2)\left[\left(0 + \frac{0.2}{2}\right)\left(0 - \frac{0.2}{2}\right) - \left(1 + \frac{0}{2}\right)\right]$=-0.1998

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$=$h\left[z_0 + \frac{l_2}{2}\right] = (0.2)\left(0 - \frac{0.1998}{2}\right) = -0.01998$

$l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = h\left[\left(x_0 + \frac{h}{2}\right)\left(z_0 + \frac{l_2}{2}\right)^2 - \left(y_0 + \frac{k_2}{2}\right)^2\right]$

$= (0.2)[(0.2)(0 - 0.1958) - (1 - 0.01998)^2] = -0.1906$

$\Delta y_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}[0 + 2(-0.02) + 2(-0.01998) - 0.0392] = -0.0199$

$\boldsymbol{y(0.2) = 0.9801}$

Exercises

1. Solve the equation $y'' + y = 0$ with the conditions y(0)=1 and $y'(0) = 0$. Compute $y(0.2)$ using R-K method.

Solution: $y(0.2) = 1.0204$

2. Find $y(0.1)$ from $\frac{d^2y}{dx^2} - y^3 = 0; y(0) = 10, y'(0) = 50$ using R-K method

Solution: $y(0.1) = 17.42$

PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations occur frequently in many branches of applied mathematics, for example in fluid dynamics, heat transfer, elasticity, quantum mechanics and electromagnetic theory. Many of these equations cannot be solved by analytical methods in closed for, solution. Since analytical solution are not available several numerical have been proposed for the solution of partial differential equations. Of all the numerical methods available, the finite difference method is most commonly used. In this method the derivatives appearing the equation are replaced by finite differences and the resulting system of algebraic equations are solved by efficient algorithms. This method was first used by L.F.Richardson and it was later improved by H.Leibmann. By doing this we get the solution at the pivotal points called numerical solution.

4.8. Classification of Partial Differential Equations of the Second Order

The general second order linear partial differential equation in two independent variables is of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$$

Which can be written as

$Au_{xx}+Bu_{xy}+Cu_{yy}+F(x,y,u,u_x,u_y)=0$ ------------------------------------ (1)

Where A,B,C,D,E,F are all functions of x and y.

A partial differential equation of the form (1) is said to be

(i) Elliptic if $B^2-4AC<0$ at a point in the (x,y) Plane.(Laplace equation)

(ii) Parabolic if $B^2-4AC=0$ at a point in the (x,y) Plane.(Heat equation)

(iii) Hyperbolic if $B^2-4AC>0$ at a point in the (x,y) Plane.(Wave equation)

Ex.1 Consider $u_{xx}+4u_{xy}+4u_{yy}- u_x+2u_y=0$

Here $B^2-4AC = 16-16 =0$ hence it is a parabolic equation.

Ex.2 Consider $x^2u_{xx}+(1-y^2)u_{yy}=0$ $-\infty<x<\infty$, $-1<y<1$

Here $B^2-4AC=0^2-4x^2(1-y^2)<0$, since $y^2<1$

Hence it is an elliptic equation

Ex.3 $(1+x^2)\,u_{xx}+(5+2x^2)^2\,u_{xy}+(4+x^2)\,u_{yy}=0$

$B^2-4AC=(5+2x^2)^2-4(1+x^2)(4+x^2)=9>0$ hence it is Hyperbolic

Ex.4 $u_{xx}+2xu_{xy}+(1-y^2)u_{yy}=0$

Here $B^2-4AC=(2x)^2-4(1-y^2)=4(x^2+y^2-1)$?

Note: The same differential equation may be elliptic in one region parabolic in another and hyperbolic in some other region. For example, the equation $xu_{xx}+u_{yy}=0$ is elliptic if $x > 0$, hyperbolic if $x < 0$ and parabolic if $x = 0$

Consider the circle $x^2+y^2=1$

(i) It is elliptic in the inside of unit circle
(ii) It is parabolic on the unit circle
(iii) It is hyperbolic outside of unit circle.

EXAMPLES

1. Classify the following partial differential equations

(i) $f_{xx}+2f_{xy}+4f_{yy}=0$ (ii) $f_{xx}-2f_{xy}+f_{yy}=0$

Solution (i) comparing this equation with (1) above, we find that

A=1, B=2, C=4

$B^2-4AC = 4-4\times1\times4 = -12<0$

Hence the equation is elliptic.

(ii) Here A=1, B =-2, C=1

$B^2-4AC = 4-4 = 0$

Hence the equation is parabolic.

Example 2 Determine whether the following equation is elliptic or hyperbolic?

$(x+1)\, u_{xx} - 2(x+2)\, u_{xy} + (x+3)\, u_{yy} = 0$

Solution: Comparing the given equation with (1) above we find that

$A = x+1$, $B = -2(x+2)$, $C = x+3$

$B^2 - 4AC = 4(x+2)^2 - 4\,(x+1)\,(x+3)$

$= 4[x2+4x+4) - (x^2+4x+3)] = 4(1) = 4 > 0$

Hence the equation is hyperbolic at all points of the region.

Example 3: Classify the following equations

(i) $x^2 \frac{\partial^2 u}{\partial x^2} + (1 - y^2) \frac{\partial^2 u}{\partial y^2} = 0, -\infty < x < \infty, -1 < y < 1$

(ii) $\frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + (x^2 + 4y^2) \frac{\partial^2 u}{\partial y^2} = \sin(x + y)$

Solution: (i) comparing the given equation with (1) above we have

$A = x^2$, $B=0$, $C=1-y^2$

$B^2 - 4AC = 0 - 4x^2(1 - y^2) = 4x^2(y^2 - 1)$

For all x between $-\infty$ and ∞, x^2 is positive.

For all y between -1 and 1, y^2-1 is negative

$B^2-4AC < 0$ if $-1<y<1, x\neq 0$

Hence for $-\infty < x < \infty$, $x\neq 0$, $-1<y<1$ the equation is elliptic.

Note: 1 $for -\infty < x < \infty, \; x \neq 0, y < -1 \; or \; y \; 1 \; the \; equation \; is \; hyperbolic.$

2. for x= 0 for all y or for all x, y = ±1 the equation is parabolic.

$(ii) Here \; A = 1, B = 4, C = x^2 + 4y^2$

$B^2 - 4AC = (4)^2 - 4.1(x^2 - 4y^2) = 4(4 - x^2 - 4y^2)$

$(a) \; The \; equation \; is \; elliptic \; if \; B^2 - 4AC < 0$

$i.e \; if \; 4 - x^2 - 4y^2 < 0$

$$If x^2 + 4y^2 > 4 \; or \; if \; \frac{x^2}{4} + \frac{y^2}{1} > 1$$

So it is elliptic outside the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

(b) The equation is parabolic if $B^2 - 4AC = 0$

i.e., if $4 - x^2 - 4y^2 = 0 \; or \; if \frac{x^2}{4} + \frac{y^2}{1} = 1$

so it is parabolic on the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

(c) The equation is hyperbolic if $B^2 - 4AC > 0$

$i.e \; if \; 4 - x^2 - 4y^2 > 0$

$i.e \; if \; 4 > x^2 + 4y^2 \; or \; \frac{x^2}{4} + \frac{y^2}{1} < 1$

So it is hyperbolic inside the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

Exercises:

Classify the following equations

1.$f_{xx} + 2f_{xy} + f_{yy} = 0$ 2.$u_{xx} + 4u_{xy} + 4u_{yy} - u_x + 2u_y = 0$

3.$(1 + x^2)f_{xx} + (5 + 2x^2)f_{xy} + (4 + x^2)f_{yy} = 0$ 4.$xu_{xx} + yu_{yy} = 0, x > 0, y > 0$

Solutions:

1. Parabolic 2. Parabolic 3. Hyperbolic 4. Elliptic

4.9. Finite Difference Approximations to Partial Derivatives

Consider a rectangular region R in the (x,y) plane. Divide this region into smaller rectangles of sides Δx = h and Δy = k by drawing the sets of lines x_i= ih, y_j = jk, i,j =0,1,2…..The points of intersection of these lines are called Pivotal points or mesh points or grid points or lattice points.

		(x,y+k)			
(x-2h,y)	(x-h,y)	(x,y)	(x+h,y)	(x+2h,y)	
(i-2,j)	(i-1,j)	(i,j)	(i+1,j)	(i+2,j)	
		(x,y-k)			
		(i,j-1) (x,y-2k)			
		(i,j-2)			

Δy = k

Δx=h

Fig. Coordinates of grid points

Notation:$u(x_i, y_i) = u_{i,j}$ mass numerical value of the solution at =(x_i, y_i)

h = step size in x-direction

k=step size in y-direction

If (x_i, y_i) is any grid $x_i = x_o + ih, y_j = y_o + jk$

Here (x=ih, y = jk) is denoted by (i,j)

Derivation of Finite difference approximations for partial derivatives

By taylor series expansion for a function of two variables

$u_{i+1,j} = u(x_{i+1,}y_j) = u(x_i + h, y_j)$

$$u_{i+1,j} = u_{i,j} + h\left(\frac{\partial u}{\partial x}\right)_{i,j} + \frac{h^2}{2!}\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} + \frac{h^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} + \frac{h^4}{4!}\left(\frac{\partial^4 u}{\partial x^4}\right)_{i,j} + \cdots (1)$$

$u_{i+1,j} = u(x_{i-1,}y_j) = u(x_i - h, y_j)$

$$u_{i+1,j} = u_{i,j} - h\left(\frac{\partial u}{\partial x}\right)_{i,j} + \frac{h^2}{2!}\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} - \frac{h^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} + \frac{h^4}{4!}\left(\frac{\partial^4 u}{\partial x^4}\right)_{i,j} + \cdots (2)$$

From (1)

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j-u_{i,j}}}{h} - \frac{h}{2!}\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} + \cdots \ldots$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j-u_{i,j}}}{h} + O(h)$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} \cong \frac{u_{i+1,j-u_{i,j}}}{h} \ldots\ldots (3)$$

Is called first order finite difference(Forward) approximation.

Further we have from (2)

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} \cong \frac{u_{i+1,j-u_{i,j}}}{h} \ldots\ldots(4)$$

Which is also first order finite difference(Backward approximation)

Equation 1-2 will give us

$$u_{i+1,j} - u_{i-1,j} = 2h\left(\frac{\partial u}{\partial x}\right)_{i,j} + \frac{2h^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} ..$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j-u_{i,j}}}{2h} + O(h^2)$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} \cong \frac{u_{i+1,j-u_{i,j}}}{2h}$$

is called second order finite difference(central) approximation.

Equations (1) + (2) will gibe us,

$$u_{i+1,j} + u_{i-1,j} = 2u_{i,j} + \frac{2h^2}{2!}\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} + \frac{2h^4}{4!}\left(\frac{\partial^4 u}{\partial x^4}\right)_{i,j}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i-1,j} + u_{i+1,j}}{h^2} + O(h^2)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} \cong \frac{u_{i+1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

Is called second order finite difference approximation(central)

Similarly, we can write

$$u_{i,j+1} = u(x_i, y_{j+1}) = u(x_i, y_{j+k})$$

$$u_{i,j+1} = u_{i,j} + k\left(\frac{\partial u}{\partial x}\right)_{i,j} + \frac{k^2}{2!}\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} + \frac{k^3}{3!}\left(\frac{\partial^3 u}{\partial y^3}\right)_{i,j} + \frac{k^4}{4!}\left(\frac{\partial^4 u}{\partial y^4}\right)_{i,j} +..$$

$$u_{i,j-1} = u_{i,j} - k\left(\frac{\partial u}{\partial x}\right)_{i,j} + \frac{k^2}{2!}\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} - \frac{k^3}{3!}\left(\frac{\partial^3 u}{\partial y^3}\right)_{i,j} + \frac{k^4}{4!}\left(\frac{\partial^4 u}{\partial y^4}\right)_{i,j} +..$$

accordingly we get

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} \cong \frac{u_{i,j+1} - u_{i,j}}{k}$$

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} \cong \frac{u_{i,j} - u_{i,j-1}}{k}$$

As first order approximations error 0(k). By substracting we get

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} \cong \frac{u_{i,j} - u_{i,j-1}}{k}$$

As the second order finite difference approximation error $O(k^2)$

By adding we get

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} \cong \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

As second order finite difference approximation with error $O(k^2)$

Further we observe that if we have time variable t, we get for $u_t(x,t)$ as

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} \cong \frac{u_{i,j+1} - u_{i,j}}{k}$$

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} \cong \frac{u_{i,j} - u_{i,j-1}}{k}$$

Are first order approximations with error O(k) and

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} \cong \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

As the second order approximation with error $O(k^2)$

Writing u(x,y)=u (ih,jk) as simply $u_{i,j}$ the above approximations become

$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h} + O(h) \dots \dots (1)$$

$$u_x = \frac{u_{i,j} - u_{i-1,j}}{h} + O(h) \dots \dots (2)$$

$$u_x = \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2) \dots \dots \dots (3)$$

$$and\ u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(h^2) \dots \dots (4)$$

Similarly we have the approximations

$$u_y = \frac{u_{i,j+1} - u_{i,j}}{k} + O(k) \dots \dots \dots (5)$$

$$u_y = \frac{u_{i,j} - u_{i,j-1}}{k} + O(k) \dots \dots (6)$$

$$u_y = \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2) \dots \dots (7)$$

$$and\ u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + O(k^2) \dots \dots (8)$$

We can now obtain the finite difference analogues of partial differential equations by replacing the derivatives in any equation by their corresponding difference equations (1) to (8).

Definitions:

Order of finite difference method: the minimum order of finite difference approximation used is called the Order of finite difference method.

Explict method: if a solution at (j+1) stage is obtained using solution upto (j) stage, we say explicit method and we get explicit solution.

Implict method: if solution at (j+1) stage requires solutions upto (j+1) stage we say implicit method. On applying this implicit method we get system of algebraic equations which will produce the solution of (j+1) stage.

Elliptic Equations

The laplace Equations

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ldots \ldots (1)$$

Or $u_{xx} + u_{yy} = 0 \ldots \ldots (2)$

Are examples of elliptic partial differential equations

Laplace equation arise in steady state flow and potential problems. Poisson's equation arises in fluid mechanics, electricity and magnetism.

Consider a rectangular region R for which u(x,y) is known as the boundary. Divide this region into a network of small squares of side h. replacing the derivatives in (1) by their difference approximations we have

$$\frac{1}{h^2}\left[u_{i+j,j} - u_{i-j,j}\right] + \frac{1}{k^2}\left[u_{i,j+1} - 2u_{i,j} + u_{i,j} - 1\right] = 0$$

If h=k this gives

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right] \ldots .. (3)$$

This shows that the value of u at any interior mesh point is the average of its four nearest neighbours. That is each u value at any interior point is the average of the values of u at four neighbouring points to the left, right, above and below. Equ (3) called the standard five point formula (SFPF)

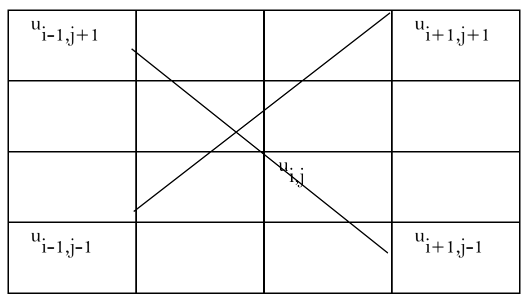
		$u_{i,j+1}$	
	$u_{i-1,j}$	$u_{i,j}$	$u_{i+1,j}$
		$u_{i,j-1}$	

Diagonal Five Point Formula (DFPF)

Sometimes a formula similar to (3) is used which is given by

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}\right] \ldots\ldots (4)$$

This shows that the value of $u_{i,j}$ is the average of its values at the four neightbouring diagonal mesh points. Equ (4) called the diagonal five-point formula.



4.10. SOLUTION OF LAPLACE's EQUATION (By Leibmann's Iterative Method)

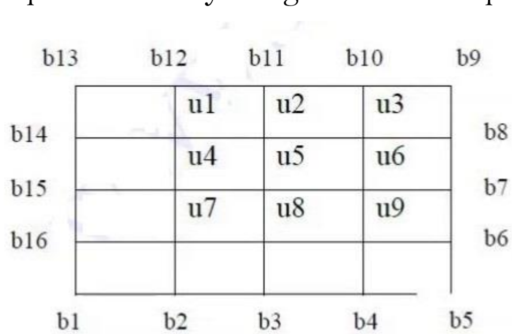
We wish to solve the Laplace's equation

$$u_{xx} + u_{yy} = 0 \ldots\ldots (1)$$

In a bounded region R with boundary C

Also let the value of u be specified everywhere on C (or atleast at the grid points in the boundary). For simplicity let R be a square region so that it can be divided into a network of small squares of side h. Let the values of u(x,y) on the boundary C be given by $b_1, b_2, \ldots b_{16}$. Also let the values of u at the interior mesh points or grid points be $u_1, u_2, \ldots u_9$

To start the iteration process, we first compute rough values at the interior mesh-points and then we improve them by iterative process mostly using standard five point formula.



Find u_5 first: $u_5 = \frac{1}{4}(b_3 + b_1 + b_{11} + b_{15})$ using SFPF

Knowing u_5 we find $u_1 + u_3 + u_7 + u_9$ (values at the center of the four larger inner squares) by using diagonal five point formula

$u_1 = \frac{1}{4}(b_3 + b_{15} + b_1 + u_5)$ using DFPF

$$u_3 = \frac{1}{4}(b_5 + u_5 + b_3 + b_7)$$

$$u_7 = \frac{1}{4}(u_5 + b_{13} + b_{11} + b_{15})$$

$$u_9 = \frac{1}{4}(b_1 + b_{11} + b_9 + u_5)$$

The remaining four values $u_2 + u_4 + u_6 + u_8$ are completed by using SFPF

$$u_2 = \frac{1}{4}(b_3 + u_5 + u_1 + u_3)$$

$$u_4 = \frac{1}{4}(u_1 + u_7 + u_5 + b_{15})$$

$$u_6 = \frac{1}{4}(u_3 + u_9 + u_5 + b_7)$$

$$u_8 = \frac{1}{4}(u_5 + b_{11} + u_7 + u_9)$$

Having found all the nine values of $u_{i,j}$ once their accuracy is imporved by using the iterative formula to solve (3) given by

$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{n+1} + u_{i+1,j}^{1} + u_{i,j+1}^{n+1} + u_{i,j-1}^{n}\right]$……(A)

Where the superscript u denotes the iteration number.

It utilises the latest iterative value available values of u to use in the formula (A).

Equation (A) is called Leibmann's iteration process. The process is repeated till the difference between two consecutive iterates becomes negligible.

Find the values of u(x,y) satisfying the Laplace's equation $\nabla^2 u = 0$ at the pivotal points of a square region, with boundary values as shown in the following figure.

Let $u_{1,}u_2 \dots u_9$ be the values of u at the interest mesh points or grid points. We will first find rough values of u and then proceed to refine them

Finding intial values:

$u_5 = \frac{1}{4}\,(0 + 17 + 21 + 12.1) = 12.5$ (SFPF)

$u_1 = \frac{1}{4}\,(0 + u_5 + 0 + 17)$ (DFPF)

$= \frac{1}{4}\,(12.5 + 17.0) = 7.4$

$u_3 = \frac{1}{4}\,(u_5 + 18.6 + 17 + 21) = \frac{1}{4}\,(12.5 + 18.6 + 17 + 21) = \ 17.3$ (DFPF)

$u_7 = \frac{1}{4}\,(0 + u_5 + 0 + 12.1) = \frac{1}{4}\,(0 + 12.5 + 0 + 12.1) = \ 6.2$

$u_9 = \frac{1}{4}\,(12.1 + 21.0 + u_5 + 9.0) = \frac{1}{4}\,(12.1 + 21.0 + 12.5 + 9.0) = \ 13.7$

$$u_2 = \frac{1}{4}\,(17.0 + u_5 + u_1 + u_3) = \frac{1}{4}\,(17.0 + 12.5 + 7.4 + 17.3) = \ 13.6$$

$$u_4 = \frac{1}{4}\,(u_1 + u_7 + 0 + u_5) = \frac{1}{4}\,(7.4 + 6.2 + 0 + 12.5) = \ 6.5$$

$$u_6 = \frac{1}{4}\,(12.5 + 21.0 + 17.3 + 13.7) = \ 16.1$$

$$u_8 = \frac{1}{4}\,(12.5 + 12.1 + 6.2 + 13.7) = \ 11.1$$

Now we have got the rough values at all interior mesh-points and already we possess the boundary values. We will now improve the values by iteration process using SFPF taking into account the latest available values of u.

$$u_1^{(n+1)} = \frac{1}{4}\left[0 + u_2^{(n)} + 11.1 + u_4^{(n)}\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[u_1^{(n+1)} + u_3^{(n)} + 17.0 + u_5^{(n)}\right]$$

$$u_3^{(n+1)} = \frac{1}{4}\left[u_2^{(n)} + 21.9 + 19.7 + u_6^{(n)}\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[0 + u_5^{(n)} + u_1^{(n+1)} + u_7^{(n)}\right]$$

$$u_5^{(n+1)} = \frac{1}{4}\left[u_4^{(n+1)} + u_6^{(n)} + u_2^{(n+1)} + u_8^{(n)}\right]$$

$$u_6^{(n+1)} = \frac{1}{4}\left[u_5^{(n+1)} + 21 + u_3^{(n+1)} + u_9^{(n)}\right]$$

$$u_7^{(n+1)} = \frac{1}{4}\left[0 + u_8^{(n)} + u_4^{(n+1)} + 8.7\right]$$

$$u_8^{(n+1)} = \frac{1}{4}\left[u_7^{(n+1)} + u_9^{(n)} + u_5^{(n+1)} + 12.1\right]$$

$$u_9^{(n+1)} = \frac{1}{4}\left[u_8^{(n+1)} + 17.0 + u_6^{(n+1)} + 12.8\right]$$

First Iteration(put n=0)

$$u_1^{(1)} = \frac{1}{4}\left[0 + u_2^{(0)} + 11.1 + u_4^{(0)}\right] = \frac{1}{4}[0 + u_2 + 11.1 + u_4] = \frac{1}{4}[0 + 13.6 + 11.6 + 6.5]$$

$$= 7.8$$

$$u_2^{(1)} = \frac{1}{4}\left[u_1^{(1)} + u_3 + 17.0 + u_5^{(1)}\right] = 13.7$$

$$u_3^{(1)} = \frac{1}{4}\left[u_2^{(1)} + 21.9 + 19.7 + u_6^{(1)}\right] = 17.9$$

$$u_4^{(1)} = \frac{1}{4}\left[0 + u_5^{(1)} + u_1^{(1)} + u_7^{(1)}\right] = 6.6$$

$$u_5^{(n+1)} = \frac{1}{4}\left[u_4^{(1)} + u_6^{(1)} + u_2^{(1)} + u_8^{(1)}\right] = 11.9$$

$$u_6^{(1)} = \frac{1}{4}\left[u_5^{(1)} + 21 + u_3^{(1)} + u_9^{(1)}\right] = 16.1$$

$$u_7^{(1)} = \frac{1}{4}\left[0 + u_8^{(1)} + u_4^{(1)} + 8.7\right] = 6.6$$

$$u_8^{(1)} = \frac{1}{4}\left[u_7^{(1)} + u_9^{(1)} + u_5^{(1)} + 12.1\right] = 11.1$$

$$u_9^{(1)} = \frac{1}{4}\left[u_8^{(1)} + 17.0 + u_6^{(1)} + 12.8\right] = 14.3$$

Second Iteration

$$u_1^{(2)} = \frac{1}{4}[0 + 11.1 + 13.7 + 6.6] = 7.9$$

$$u_2^{(2)} = \frac{1}{4}[17.0 + 17.9 + 7.9 + 11.9] = 13.7$$

$$u_3^{(2)} = \frac{1}{4}[13.7 + 19.7 + 21.9 + 16.1] = 17.9$$

$$u_4^{(2)} = \frac{1}{4}[7.9 + 0 + 11.9 + 6.6] = 6.6$$

$$u_5^{(2)} = \frac{1}{4}[13.7 + 6.6 + 16.1 + 11.1] = 11.9$$

$$u_6^{(2)} = \frac{1}{4}[11.9 + 17.9 + 21.0 + 14.3] = 16.3$$

$$u_7^{(2)} = \frac{1}{4}[0 + 6.6 + 11.1 + 8.7] = 6.6$$

$$u_8^{(2)} = \frac{1}{4}[6.6 + 11.9 + 14.3 + 12.1] = 11.2$$

$$u_9^{(2)} = \frac{1}{4}[11.2 + 16.3 + 17.0 + 12.8] = 14.3$$

Third Iteration (Put n=2)

$$u_1^{(3)} = \frac{1}{4}[0 + 11.1 + 13.7 + 6.6] = 7.9$$

$$u_2^{(3)} = \frac{1}{4}[17.0 + 17.9 + 7.9 + 11.9] = 13.7$$

$$u_3^{(3)} = \frac{1}{4}[13.7 + 19.7 + 21.9 + 16.1] = 17.9$$

$$u_4^{(3)} = \frac{1}{4}[7.9 + 0 + 11.9 + 6.6] = 6.6$$

$$u_5^{(3)} = \frac{1}{4}[13.7 + 6.6 + 16.1 + 11.1] = 11.9$$

$$u_6^{(3)} = \frac{1}{4}[11.9 + 17.9 + 21.0 + 14.3] = 16.3$$

$$u_7^{(3)} = \frac{1}{4}[0 + 6.6 + 11.1 + 8.7] = 6.6$$

$$u_8^{(3)} = \frac{1}{4}[6.6 + 11.9 + 14.3 + 12.1] = 11.2$$

$$u_9^{(3)} = \frac{1}{4}[11.2 + 16.3 + 17.0 + 12.8] = 14.3$$

Since the values obtained in the second and third iterations are same, we stop the procedure.

Hence $u_1 = 7.9, u_2 = 13.7, u_3 = 17.9, u_4 = 6.6, u_5 = 11.9, u_6 = 16.3, u_7 = 6.6, u_8 = 11.2, u_9 = 14.3$

Example 2: Evaluate the function u(x,y) satisfying $\nabla^2 u = 0$ at the pivotal points given the boundary values as follows:

Let $u_{1,} u_{2,} u_{3,} u_{4,}$ be the values of u at the interest mesh points or grid points.

To get the initial values of $u_{1,} u_{2,} u_{3,} u_{4,}$ we assume that $u_4 = 0$ (or any other u)

Then

$u_1 = \frac{1}{4}\ (1000 + 0 + 1000 + 2000) = 1000$ (DFPF)

$u_2 = \frac{1}{4}\ (1000 + 500 + 1000 + 0) = 625$ (SFPF)

$u_3 = \frac{1}{4}\ (2000 + 0 + 1000 + 500) = 875$ (SFPF)

$u_4 = \frac{1}{4}\ (875 + 0 + 625 + 0) = 375$ (SFPF)

We carry out the iteration process using the formulae

$$u_1^{(n+1)} = \frac{1}{4}\left[2000 + u_2^{(n)} + 1000 + u_3^{(n)}\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[u_1^{(n+1)} + 500 + 1000 + u_4^{(n)}\right]$$

$$u_3^{(n+1)} = \frac{1}{4}\left[2000 + u_4^{(n)} + u_1^{(n+1)} + 500\right]$$

$$u_4^{(n+1)} = \frac{1}{4}\left[u_3^{(n+1)} + 0 + u_2^{(n+1)} + 0\right]$$

First iteration (put n=0)

$$u_1^{(1)} = \frac{1}{4}[2000 + 625 + 1000 + 875] = 1125$$

$$u_2^{(1)} = \frac{1}{4}[1125 + 500 + 1000 + 375] = 750$$

$$u_3^{(1)} = \frac{1}{4}[2000 + 375 + 1125 + 500] = 1000$$

$$u_4^{(1)} = \frac{1}{4}[1000 + 0 + 750 + 0] = 438$$

Second iteration (put n=1)

$$u_1^{(2)} = \frac{1}{4}[2000 + 750 + 1000 + 1000] = 1188$$

$u_2^{(2)} = \frac{1}{4}[1188 + 500 + 1000 + 438] = 782$

$u_3^{(2)} = \frac{1}{4}[2000 + 438 + 1188 + 500] = 1032$

$u_4^{(2)} = \frac{1}{4}[1032 + 0 + 782 + 0] = 454$

Similarly

$u_1^{(3)} = 1204, u_2^{(3)} = 789, u_3^{(3)} = 1040, u_4^{(3)} = 458$

$u_1^{(4)} = 1207, u_2^{(4)} = 791, u_3^{(4)} = 1041, u_4^{(4)} = 458$

$u_1^{(5)} = 1208, u_2^{(5)} = 791.5, u_3^{(5)} = 1041.5, u_4^{(5)} = 458.25$

Thus there is negligible difference between the values obtained in the fourth and fifth iterations.

Hence $u_1 = 1208, u_2 = 792, u_3 = 1042, u_4 = 458$

REVIEW QUESTIONS

1. Write the diagonal five point-formula to solve the Laplace equation $u_{xx} + u_{yy} = 0$ and explain the procedure to solve it.

4.11. Solutions of Poisson's Equation

Poisson's Equation

If we replace $\underline{E}$ with V in the differential form of Gauss's Law we get **Poisson's Equa-tion**:

$$?^2 V = \frac{\rho}{0}$$

Where the Laplacian operator reads in Cartesians $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$

It relates the second derivatives of the potential to the local charge density.

In a region absent of free charges it reduces to **Laplace's equation**:

$$\nabla^2 V = 0$$

Note that one solution is a uniform potential $V = V_0$, but this would only apply to the case where there are no free charges anywhere. More generally we have to solve Laplace's equation subject to certain *boundary conditions* and this yields non-trivial solutions.

Poisson's and Laplace's equations are among the most important equations in physics, not just EM: fluid mechanics, diffusion, heat flow etc. They can be studied using the techniques you have seen Physical Mathematics e.g. separation of variables, orthogonal polynomials etc.,

4.12. GAUSS SEIDAL ITERATION METHOD

This is a modification of Gauss Jacobi method.

We will consider the system of equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \quad 1$$

Where the diagonal coefficients are not zero and are large compared to other coefficients. Such a system is called a diagonally dominant system.

The system of equ(1) may be written as

$$\mathbf{x_1} = \frac{\mathbf{1}}{\mathbf{a_{11}}}[\mathbf{b_1} - \mathbf{a_{12}x_2} - \mathbf{a_{13}x_3}] \quad 2$$

$$\mathbf{x_2 = \frac{1}{a_{22}}[b_2 - a_{21}x_1 - a_{23}x_3]}$$

$$\mathbf{x_3 = \frac{1}{a_{33}}[b_3 - a_{31}x_1 - a_{32}x_2]}$$

Let the initial approximate solution be $\boldsymbol{x_1^{(0)}, x_2^{(0)}, x_3^{(0)}}$. Substituting $\boldsymbol{x_2^{(0)}, x_3^{(0)}}$ in the first equation of (2) we get

$\mathbf{x_2^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right]}$ ....3(a)

This is taken as the first approximation of x_1

Substituting $\mathbf{x_1^{(1)}}$ **for** $\mathbf{x_1}$ **and** $\mathbf{x_3^{(0)}}$ **for** $\mathbf{x_3^{(0)}}$ in the second equation of (2) we get

$\mathbf{x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}\right]}$..........3(b)

This is taken as the first approximation of x_2.

Next substituting $\mathbf{x_1^{(1)}}$ **for** $\mathbf{x_1}$ **and** $\mathbf{x_2^{(1)}}$ **for** $\mathbf{x_2}$ in the last equation of (2) we get

$\mathbf{x_3^{(1)} = \frac{1}{a_{22}}\left[b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right]}$.......3(c)

This is taken as the first approximation of x_3.

The values of obtained in 3(a),3(b),3(c) constitute the first iterates of the solution.

Proceeding in the same way, we get successive iterates.

The (k+1) iterates are given by

$$\boldsymbol{x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right]}$$

$$\boldsymbol{x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}\right]}$$

$$\boldsymbol{x_3^{(k+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}\right]}$$

The iteration process is stopped when the desired order of approximation is reached or two successive iterations are nearly the same. The final values of x_1, x_2, x_3 so obtained constitute an approximate solution of the system(2)

This method can be generalized to a system of n equations n unknowns. The method is known as Gauss-Seidal iteration method. This method is also called method of successive displacement.

Example 1: Use Gauss Seidal iteration method to solve the system.

10x+y+z=12

2x+10y+z=13

2x+2y+10z=14

Solution: The given system is diagonally dominant and we write it as

$x = \frac{1}{10}[12 - y - z]$…….(1)

$y = \frac{1}{10}[13 - 2x - z]$…….(2)

$z = \frac{1}{10}[14 - 2x - 2y]$……(3)

We start iteration by taking y=0,z=0 in (1) we get

$$x^{(1)} = 1.2$$

Putting $x = x^1 = 1.2, z = 0 \; in \; (2)$ we get

$$y^{(1)} = 1.06$$

Putting $x = 1.2, y = 1.06 \; in \; (3)$ we get

$$z^{(1)} = 0.95$$

Now taking the $y^{(1)}$, $z^{(1)}$as the initial values in (1) we get

$x^{(2)} = \frac{1}{10}[12 - 1.06 - 0.95]$=0.999

taking the $\boldsymbol{x = x^{(2)}}$, $\boldsymbol{z = z^{(1)}\ in\ (2)}$ we get

$\boldsymbol{y^{(2)} = \frac{1}{10}[13 - 1.998 - 0.95]}$=1.005

taking the $\boldsymbol{x = x^{(2)}}$, $\boldsymbol{y = y^{(2)}\ in\ (3)}$ we get

$\boldsymbol{z^{(2)} = \frac{1}{10}[14 - 1.998 - 2.010]}$=0.999

Again taking the $\boldsymbol{x^{(2)}, y^{(2)}, z^{(2)}}$ as the initial values we get

$\boldsymbol{x^{(3)} = \frac{1}{10}[12 - 1.005 - 0.999]}$=0.996=1.00

$\boldsymbol{y^{(3)} = \frac{1}{10}[13 - 2.0 - 0.999]}$=1.0001=1.00

$\boldsymbol{z^{(3)} = \frac{1}{10}[14 - 2 - 2]}$=1.00

Similarly we find the fourth approximation of x,y,z and get them as $\boldsymbol{x^{(4)} = 1.00, y^{(4)} = 1.00, z^{(4)} = 1.00}$

The solution of the equation is x=1,y=1,z=1

Example2: Solve the following systems of equations by Gauss seidal method

$$\boldsymbol{8x_1 - 3x_2 + 2x_3 = 20, 4x_1 + 11x_2 - x_3 = 20, 6x_1 + 3x_2 + 12x_3 = 36}$$

The given system is diagonally dominant and we write it as

$$\boldsymbol{x_1 = \frac{1}{8}(20 + 3x_2 - 2x_3)}$$

$$\boldsymbol{x_2 = \frac{1}{11}(33 - 4x_1 + x_3)}$$

$$\boldsymbol{x_3 = \frac{1}{12}(36 - 6x_1 - 3x_2)}$$

We start iteration by taking $\boldsymbol{x_2 = 0, x_3 = 0\ in\ (1) to\ get}$

$$x_1^{(1)} = \frac{1}{8} X\ 20 = 2.5$$

Putting $x_1 = 2.5, x_3 = 0\ in\ (2) to\ get$

$$x_2^{(1)} = \frac{1}{11} X\ 23 = 2.1$$

Putting $x_1 = 2.5, x_2 = 2.1\ in\ (3) to\ get$

$$x_3^{(1)} = \frac{1}{12}(36 - 15 - 6.3) = 1.2$$

The required solution is

$$x_1 = 2.9998, x_2 = 2.0000, x_3 = 1.0000$$

EXERCISES

Using Gauss seidal method solve the following system of equations

1. 10x+2y+z=9, 2x+20y-2z=-44, -2x+3y+10z=22

2. 25x+2y+2z=69, 2x+10y+z=63,x+y+z=43

3. 20x+2y+6z=28, x+20y+9z=-23, 2x-7y-20z=-57

SOLUTIONS

1. x=1, y=-2,z=3

2. x=0.9953, y=2.116, z=39.8931

3. x=0.5149, y=-2.9451, z=3.9323

Unit -V

5.1 Program Structure

A typical C++ program consists of the following sections:

1. Preprocessor Directives:

These are commands to the compiler, starting with #, that tell it to include libraries or perform specific tasks before compiling the code.

Example: #include <iostream>

2. Namespace Declaration:

Specifies which namespace is being used, commonly std for standard library functions.

Example: using namespace std;

3. Global Declarations:

Variables or constants declared outside any function and accessible throughout the program.

Example: int globalVariable = 10;

4. Function Declarations:

Functions are declared before they are defined to inform the compiler about their return type, name, and parameters.

Example: void display();

5. Main Function:

The entry point of any C++ program where execution begins.

Example:

```
int main() {
    cout << "Hello, World!";
    return 0;
}
```

6. Function Definitions:

Contains the actual implementation of declared functions.

Example:

```
void display() {
   cout << "Function Called!";
}
```

Header Files

Header files in C++ are files with a .h extension (or standard library headers like <iostream> with no extension). They contain declarations of functions, classes, and constants that can be used in multiple files.

Types of Header Files

1. Standard Library Header Files:

Built-in headers provided by C++.

Examples:

<iostream>: For input/output operations.

<cmath>: For mathematical functions.

<vector>: For using the std::vector container.

Usage:

```
#include <iostream>
```

2. User-Defined Header Files:

Created by the programmer to reuse code across multiple programs.

Example:

myHeader.h:

```
void greet() {
    cout << "Hello from myHeader!";
}
```

Usage in program:

```
#include "myHeader.h"
```

5.2 Basic Data Types:

Int : Stores whole numbers (integers), e.g., 1, 2, -5, 100.

Float : Stores single-precision floating-point numbers (decimals), e.g., 3.14, 2.718.

Double: Stores double-precision floating-point numbers (decimals with higher precision), e.g., 3.141592653589793.

Char : Stores single characters, e.g., 'a', 'b', 'Z'.

Bool : Stores boolean values (true or false).

Void : Represents the absence of a type.

Modified Data Types:

Signed int : Same as int, but explicitly specifies that the integer can be positive or negative.

Unsigned int : Stores only non-negative integers.

Short int : Stores smaller integers, using less memory than int.

Long int : Stores larger integers, using more memory than int.

Long long int : Stores even larger integers than long int.

5.3 Opertors:

1. Arithmetic Operators:

+: Addition, -: Subtraction, *: Multiplication, /: Division, and %: Modulus (remainder after division).

2. Assignment Operators:

=: Assignment

+=: Add and assign

-=: Subtract and assign

*=: Multiply and assign

/=: Divide and assign

%=: Modulus and assign

3. Relational Operators:

==: Equal to

!=: Not equal to

<: Less than

>: Greater than

<=: Less than or equal to

>=: Greater than or equal to

4. Logical Operators:

&&: Logical AND, ||: Logical OR, and !: Logical NOT.

5. Bitwise Operators:

&: Bitwise AND, |: Bitwise OR, ^: Bitwise XOR, ~: Bitwise NOT, <<: Left shift, and >>: Right shift.

6. Increment/Decrement Operators:

++: Increment (pre-increment, post-increment)

--: Decrement (pre-decrement, post-decrement)

7. Conditional Operator (Ternary Operator):

?:: (condition) ? (expression1) : (expression2)

8. Other Operators:

sizeof: Returns the size of a variable or type

typeid: Returns the type of a variable

::: Scope resolution operator

.*: Pointer to member operator

->: Member access operator

new: Dynamic memory allocation

delete: Dynamic memory deallocation

(): Function call operator

[]: Array subscript operator

5.4 Decision Making Statements:

if statement	:Executes a block of code if a specified condition is true.
if-else statement	:Executes one block of code if a condition is true, and another block if it's false.
if-else if-else ladder	:Allows you to check multiple conditions and execute different code blocks based on the first true condition.
switch statement	:Provides a way to select from multiple code blocks based on the value of an expression.

Example of decision making:

```
#include <iostream>
int main() {
   int num = 10;
   if (num > 0) {
```

```
        std::cout << "The number is positive." << std::endl;
    } else if (num < 0) {
        std::cout << "The number is negative." << std::endl;
    } else {
        std::cout << "The number is zero." << std::endl;
    }
    return 0;
}
```

Looping Statements:

for loop : Repeats a block of code a specific number of times.

while loop : Repeats a block of code as long as a specified condition is true.

do-while loop : Similar to a while loop, but it executes the code block at least once before checking the condition.

Example of looping:

```
#include <iostream>
int main() {
    for (int i = 1; i <= 5; i++) {
        std::cout << i << " ";
    }
    std::cout << std::endl;
    int j = 1;
    while (j <= 5) {
        std::cout << j << " ";
```

```
    j++;
  }
  std::cout << std::endl;
  return 0;
}
```

Arrays:

An array is a collection of elements of the same data type stored in contiguous memory locations.

You can access individual elements using their index, starting from 0.

Declaration: int numbers [5] = {1, 2, 3, 4, 5};

Strings:

C++ provides two ways to work with strings:

C-style strings: These are arrays of characters terminated by a null character (\0).

C++ strings (std::string): These are objects from the C++ Standard Library that provide a more convenient and safer way to work with strings.

Declaration:

C-style: char name [] = "John";

C++ string: std::string name = "John";

Structures:

A structure is a user-defined data type that groups together variables of different data types.

It allows you to create complex data structures.

```
struct Employee {
  int id;
  std::string name;
```

```
    double salary;
};
```

Pointers:

A pointer is a variable that stores the memory address of another variable.

They are essential for dynamic memory allocation and working with complex data structures.

Declaration:

```
int *ptr;
int x = 10; ptr = &x;
```

File Handling:

C++ provides file stream classes (ifstream, ofstream, and fstream) for reading from and writing to files.

You can open files in various modes (read, write, append) and perform operations like reading, writing, and seeking.

Example:

```
#include <iostream>
#include <fstream>
int main() {
    std::ofstream outfile("data.txt");
    outfile << "Hello, World!" << std::endl;
    outfile.close();
    std::ifstream infile("data.txt");
    std::string line;
    while (std::getline(infile, line)) {
        std::cout << line << std::endl;
```

```
    }
    infile.close();
    return 0;
}
```

5.5 Newton-Raphson Method to find the root of a given algebraic or transcendental equation.

```
#include <iostream>
#include <cmath>
#include <iomanip>
// Define the function for the equation
double f(double x) {
    // Example: Change as needed (e.g., x^3 - x - 1 for algebraic or cos(x) - x*exp(x) for
                    transcendental)
    return x * x * x - x - 1; // Example: x^3 - x - 1
}
// Define the derivative of the function
double f_prime(double x) {
    // Derivative of the function above
    return 3 * x * x - 1; // Example: 3x^2 - 1
}
// Newton-Raphson method
void newton_raphson(double initial_guess, double tolerance = 1e-6, int max_iterations = 100)
                    {
    double x = initial_guess;
```

```
    int iteration = 0;
    double error = 1.0;
    std::cout << std::fixed << std::setprecision(6);
    std::cout << "Iteration\tX\t\tf(X)\t\tError\n";
    while (iteration < max_iterations && error > tolerance) {
        double fx = f(x);
        double fpx = f_prime(x);
        if (std::abs(fpx) < 1e-10) { // Avoid division by zero
            std::cout << "Derivative is too small; method fails.\n";
            return;
        }
        double x_new = x - fx / fpx;
        error = std::abs(x_new - x);
        std::cout << iteration + 1 << "\t\t" << x_new << "\t" << fx << "\t" << error << "\n";
        x = x_new;
        iteration++;
    }
    if (error <= tolerance) {
        std::cout << "\nRoot found: " << x << " after " << iteration << " iterations.\n";
    } else {
        std::cout << "\nMethod failed to converge after " << max_iterations << " iterations.\n";
    }
}
```

```
// Main function
int main() {
    double initial_guess;
    // Example algebraic equation: x^3 - x - 1
    // Example transcendental equation: cos(x) - x*exp(x)
    std::cout << "Solving equation x^3 - x - 1 using Newton-Raphson method\n";
    std::cout << "Enter initial guess: ";
    std::cin >> initial_guess;
    newton_raphson(initial_guess);
    return 0;
}
```

Input

Enter initial guess: 1.5

Output

Solving equation x^3 - x - 1 using Newton-Raphson method

Enter initial guess: 1.5

Iteration	X	f(X)	Error
1	1.347826	0.875000	0.152174
2	1.325201	0.015935	0.022625
3	1.324718	0.000011	0.000483
4	1.324718	0.000000	0.000000

Root found: 1.324718 after 4 iterations.

5.6 Charging And Discharging of a Condenser By Euler's Method

```
#include <iostream>

#include <cmath>

#include <iomanip>

// Function for charging of a capacitor

double charging(double q, double R, double C, double V) {

   return (V / R) - (q / (R * C)); // dq/dt = (V/R) - (q / (R*C))

}

// Function for discharging of a capacitor

double discharging(double q, double R, double C) {

   return -(q / (R * C)); // dq/dt = -(q / (R*C))

}

// Euler's Method Implementation

void euler_method(bool is_charging, double R, double C, double V, double q0, double dt, double
                        t_max) {

   double t = 0.0;  // Start time

   double q = q0;   // Initial charge on the capacitor

   double dq_dt;    // Rate of change of charge

   std::cout << std::fixed << std::setprecision(6);

   std::cout << "Time(s)\tCharge(C)\n";

   while (t <= t_max) {

      std::cout << t << "\t" << q << "\n";

      // Compute dq/dt based on whether charging or discharging
```

```
        if (is_charging) {
            dq_dt = charging(q, R, C, V);
        } else {
            dq_dt = discharging(q, R, C);
        }
        // Update charge using Euler's method
        q += dq_dt * dt;
        // Increment time
        t += dt;
    }
}
// Main function
int main() {
    double R, C, V, q0, dt, t_max;
    int mode;
    std::cout << "Charging and Discharging of a Capacitor using Euler's Method\n";
    std::cout << "Enter resistance (R) in ohms: ";
    std::cin >> R;
    std::cout << "Enter capacitance (C) in farads: ";
    std::cin >> C;
    std::cout << "Enter time step (dt) in seconds: ";
    std::cin >> dt;
    std::cout << "Enter maximum simulation time (t_max) in seconds: ";
```

```
    std::cin >> t_max;
    std::cout << "Enter initial charge (q0) in coulombs: ";
    std::cin >> q0;
    std::cout << "Choose mode (1 for charging, 2 for discharging): ";
    std::cin >> mode;
    if (mode == 1) {
        std::cout << "Enter supply voltage (V) in volts: ";
        std::cin >> V;
        std::cout << "\nSimulating Charging...\n";
        euler_method(true, R, C, V, q0, dt, t_max);
    } else if (mode == 2) {
        std::cout << "\nSimulating Discharging...\n";
        euler_method(false, R, C, 0.0, q0, dt, t_max);
    } else {
        std::cerr << "Invalid mode selected. Please enter 1 or 2.\n";
    }
    return 0;
}
```

Input

Enter resistance (R) in ohms: 1000

Enter capacitance (C) in farads: 0.001

Enter time step (dt) in seconds: 0.01

Enter maximum simulation time (t_max) in seconds: 1

Enter initial charge (q0) in coulombs: 0

Choose mode (1 for charging, 2 for discharging): 1

Enter supply voltage (V) in volts: 5

Output

Simulating Charging...

Time(s) Charge(C)

0.000000 0.000000

0.010000 0.004950

0.020000 0.009801

0.030000 0.014554

0.040000 0.019208

5.7 Radioactive Decay By Runge Kutta Fourth Order Method

```
#include <iostream>

#include <cmath>

#include <iomanip>

// Define the decay rate function (dy/dt = -k * y)

double decay_rate(double t, double y, double k) {

    return -k * y;

}

// Runge-Kutta 4th Order Method Implementation

void runge_kutta_4th_order(double y0, double t0, double t_max, double dt, double k) {

    double t = t0;  // Start time

    double y = y0;  // Initial amount of substance
```

```
    double k1, k2, k3, k4;  // RK4 coefficients
    std::cout << std::fixed << std::setprecision(6);
    std::cout << "Time(s)\tAmount\n";
    while (t <= t_max) {
        std::cout << t << "\t" << y << "\n";
        // Compute RK4 coefficients
        k1 = dt * decay_rate(t, y, k);
        k2 = dt * decay_rate(t + dt / 2.0, y + k1 / 2.0, k);
        k3 = dt * decay_rate(t + dt / 2.0, y + k2 / 2.0, k);
        k4 = dt * decay_rate(t + dt, y + k3, k);
        // Update y (amount of substance) using RK4 formula
        y = y + (k1 + 2 * k2 + 2 * k3 + k4) / 6.0;
        // Increment time
        t += dt;
    }
}
int main() {
    double y0, t0, t_max, dt, k;
    std::cout << "Radioactive Decay using Runge-Kutta Fourth Order Method\n";
    std::cout << "Enter initial amount of substance (y0): ";
    std::cin >> y0;
    std::cout << "Enter initial time (t0): ";
    std::cin >> t0;
```

```
    std::cout << "Enter maximum simulation time (t_max): ";
    std::cin >> t_max;
    std::cout << "Enter time step (dt): ";
    std::cin >> dt;
    std::cout << "Enter decay constant (k): ";
    std::cin >> k;
    std::cout << "\nSimulating Radioactive Decay...\n";
    runge_kutta_4th_order(y0, t0, t_max, dt, k);
    return 0;
}
```

Input

Enter initial amount of substance (y0): 100

Enter initial time (t0): 0

Enter maximum simulation time (t_max): 10

Enter time step (dt): 1

Enter decay constant (k): 0.1

Output

Simulating Radioactive Decay...

Time(s) Amount

0.000000 100.000000

1.000000 90.483742

2.000000 81.873075

3.000000 74.081826

```
4.000000    67.032155
5.000000    60.653066
6.000000    54.879359
7.000000    49.651638
8.000000    44.915841
9.000000    40.622090
10.000000   36.724327
```

5.8 Current in Wheatstone Bridge by Gauss Elimination Method

```
#include <iostream>
#include <iomanip>
#include <vector>
using namespace std;
// Function to perform Gauss Elimination
void gauss_elimination(vector<vector<double>>& matrix, vector<double>& currents, int n) {
    // Forward Elimination
    for (int i = 0; i < n; i++) {
        // Make the diagonal element 1 and scale the row
        double diag_element = matrix[i][i];
        for (int j = 0; j <= n; j++) {
            matrix[i][j] /= diag_element;
        }
        // Eliminate below the pivot
        for (int k = i + 1; k < n; k++) {
```

```
            double factor = matrix[k][i];
            for (int j = 0; j <= n; j++) {
                matrix[k][j] -= factor * matrix[i][j];
            }
        }
    }
    // Back Substitution
    for (int i = n - 1; i >= 0; i--) {
        currents[i] = matrix[i][n];
        for (int j = i + 1; j < n; j++) {
            currents[i] -= matrix[i][j] * currents[j];
        }
    }
}
int main() {
    int n = 3; // Number of equations for the Wheatstone Bridge
    vector<vector<double>> matrix(n, vector<double>(n + 1)); // Augmented matrix
    vector<double> currents(n); // Solution vector for currents

    cout << "Current in Wheatstone Bridge by Gauss Elimination Method\n";
    // Input resistance and voltage
    cout << "Enter the augmented matrix row by row (coefficients and constants):\n";
    for (int i = 0; i < n; i++) {
```

```
        cout << "Row " << i + 1 << ": ";
        for (int j = 0; j <= n; j++) {
            cin >> matrix[i][j];
        }
    }
    // Perform Gauss Elimination
    gauss_elimination(matrix, currents, n);
    // Output the results
    cout << "\nThe currents in the branches are:\n";
    for (int i = 0; i < n; i++) {
        cout << "I" << i + 1 << " = " << fixed << setprecision(6) << currents[i] << " A\n";
    }
    return 0;
}
```

Input

100 150 0 10

0 200 250 10

0 0 50 0

Output

I1 = 0.050000 A

I2 = 0.030000 A

I3 = 0.000000 A

5.9 Cauchy's Constant by Least Square Method

```
#include <iostream>

#include <vector>

#include <iomanip>

#include <cmath>

using namespace std;

// Function to calculate Cauchy's constants using Least Squares Method

void cauchys_constant(const vector<double>& wavelengths, const vector<double>&
                  refractive_indices) {

   int n = wavelengths.size();

   if (n != refractive_indices.size() || n == 0) {

      cerr << "Error: Mismatched or empty data sets.\n";

      return;

   }

   // Variables for summation

   double sum_l_inv2 = 0.0, sum_l_inv4 = 0.0, sum_l_inv2_n = 0.0, sum_n = 0.0;

   // Compute summations

   for (int i = 0; i < n; i++) {

      double l_inv2 = 1.0 / (wavelengths[i] * wavelengths[i]); // 1/λ²

      double l_inv4 = l_inv2 * l_inv2;                    // (1/λ²)²

      sum_l_inv2 += l_inv2;

      sum_l_inv4 += l_inv4;

      sum_l_inv2_n += l_inv2 * refractive_indices[i];
```

```
        sum_n += refractive_indices[i];
    }
    // Calculate determinants for constants A and B
    double denominator = n * sum_l_inv4 - sum_l_inv2 * sum_l_inv2;
    double A = (sum_l_inv4 * sum_n - sum_l_inv2 * sum_l_inv2_n) / denominator;
    double B = (n * sum_l_inv2_n - sum_l_inv2 * sum_n) / denominator;
    // Output the results
    cout << fixed << setprecision(6);
    cout << "Cauchy's Constants:\n";
    cout << "A = " << A << "\n";
    cout << "B = " << B << "\n";
}
int main() {
    int n;
    cout << "Cauchy's Constant Calculation using Least Squares Method\n";
    cout << "Enter the number of data points: ";
    cin >> n;

    vector<double> wavelengths(n), refractive_indices(n);
    // Input data
    cout << "Enter the wavelengths (in micrometers) and refractive indices:\n";
    for (int i = 0; i < n; i++) {
        cout << "Data Point " << i + 1 << ":\n";
```

```
        cout << "  Wavelength (λ): ";
        cin >> wavelengths[i];
        cout << "  Refractive Index (n): ";
        cin >> refractive_indices[i];
    }
    // Perform the Least Squares calculation
    cauchys_constant(wavelengths, refractive_indices);
    return 0;
}
```

Input

```
Enter the number of data points: 3
Enter the wavelengths (in micrometers) and refractive indices:
Data Point 1:
  Wavelength (λ): 0.5
  Refractive Index (n): 1.6
Data Point 2:
  Wavelength (λ): 0.6
  Refractive Index (n): 1.55
Data Point 3:
  Wavelength (λ): 0.7
  Refractive Index (n): 1.53
```

Output

```
Cauchy's Constants:
A = 1.500000
B = 0.020833
```

5.10 Evaluation of Integral by Simpson’s Method

```
#include <iostream>
#include <cmath>
#include <iomanip>
using namespace std;
// Define the function to integrate
double f(double x) {
    // Example: f(x) = x^2 (you can change this to any function)
    return x * x;
}
// Simpson's 1/3 Rule for Numerical Integration
double simpsons_rule(double a, double b, int n) {
    // Ensure n is even (Simpson's 1/3 Rule requires an even number of intervals)
    if (n % 2 != 0) {
        cerr << "Error: Number of intervals (n) must be even.\n";
        return -1;
    }
    double h = (b - a) / n; // Step size
    double sum = f(a) + f(b); // Initialize with f(a) and f(b)
```

```
    // Summation for odd indices (4 * f(x_odd))
    for (int i = 1; i < n; i += 2) {
        sum += 4 * f(a + i * h);
    }
    // Summation for even indices (2 * f(x_even))
    for (int i = 2; i < n; i += 2) {
        sum += 2 * f(a + i * h);
    }
    return (h / 3.0) * sum;
}
int main() {
    double a, b;
    int n;
    cout << "Evaluation of Integral using Simpson's Rule\n";
    cout << "Enter the lower limit (a): ";
    cin >> a;
    cout << "Enter the upper limit (b): ";
    cin >> b;
    cout << "Enter the number of intervals (n, must be even): ";
    cin >> n;
    double result = simpsons_rule(a, b, n);
    if (result != -1) {
        cout << fixed << setprecision(6);
```

```
        cout << "The value of the integral is: " << result << "\n";
    }
    return 0;
}
```

Input

Enter the lower limit (a): 0

Enter the upper limit (b): 1

Enter the number of intervals (n, must be even): 4

Output

The value of the integral is: 0.333333

Evaluation of Integral by Monto Carlo Method

```
#include <iostream>
#include <cmath>
#include <cstdlib>
#include <ctime>
#include <iomanip>
using namespace std;
// Define the function to integrate
double f(double x) {
    // Example: f(x) = x^2 (you can change this to any function)
    return x * x;
}
// Monte Carlo Method for Numerical Integration
```

```
double monte_carlo_integration(double a, double b, int num_points) {
    double sum = 0.0;
    double range = b - a;
    // Seed the random number generator
    srand(static_cast<unsigned>(time(0)));
    // Generate random points and evaluate the function
    for (int i = 0; i < num_points; i++) {
        double x_random = a + static_cast<double>(rand()) / RAND_MAX * range; // Random
                point in [a, b]
        sum += f(x_random);
    }
    // Compute the integral estimate
    return (range / num_points) * sum;
}
int main() {
    double a, b;
    int num_points;
    cout << "Evaluation of Integral using Monte Carlo Method\n";
    cout << "Enter the lower limit (a): ";
    cin >> a;
    cout << "Enter the upper limit (b): ";
    cin >> b;
    cout << "Enter the number of random points to generate: ";
```

```
    cin >> num_points;
    double result = monte_carlo_integration(a, b, num_points);
    cout << fixed << setprecision(6);
    cout << "The estimated value of the integral is: " << result << "\n";
    return 0;
}
```

Input

Enter the lower limit (a): 0

Enter the upper limit (b): 1

Enter the number of random points to generate: 100000

Output

The estimated value of the integral is: 0.333258

5.11 Newton's Law of Cooling by Numerical Diffrentiation

```
#include <iostream>
#include <iomanip>
#include <cmath>
using namespace std;
// Function to compute the rate of cooling (dy/dt = -k(T - T_env))
double rate_of_cooling(double T, double T_env, double k) {
    return -k * (T - T_env);
}
// Newton's Law of Cooling using Numerical Differentiation (Euler's Method)
```

```
void newtons_law_of_cooling(double T_initial, double T_env, double k, double dt, double
                t_max) {
    double T = T_initial; // Initial temperature of the object
    double t = 0.0;       // Start time
    cout << fixed << setprecision(6);
    cout << "Time(s)\tTemperature(C)\n";
    while (t <= t_max) {
        cout << t << "\t" << T << "\n";
        // Calculate the temperature change using the rate of cooling
        double dT_dt = rate_of_cooling(T, T_env, k);
        T += dT_dt * dt; // Update temperature using Euler's Method
        // Increment time
        t += dt;
    }
}
int main() {
    double T_initial, T_env, k, dt, t_max;
    cout << "Newton's Law of Cooling using Numerical Differentiation\n";
    cout << "Enter the initial temperature of the object (T_initial in °C): ";
    cin >> T_initial;
    cout << "Enter the surrounding temperature (T_env in °C): ";
    cin >> T_env;
    cout << "Enter the cooling constant (k): ";
```

```
    cin >> k;
    cout << "Enter the time step (dt in seconds): ";
    cin >> dt;
    cout << "Enter the maximum simulation time (t_max in seconds): ";
    cin >> t_max;
    cout << "\nSimulating Newton's Law of Cooling...\n";
    newtons_law_of_cooling(T_initial, T_env, k, dt, t_max);
    return 0;
}
```

Input

Enter the initial temperature of the object (T_initial in °C): 80

Enter the surrounding temperature (T_env in °C): 25

Enter the cooling constant (k): 0.07

Enter the time step (dt in seconds): 1

Enter the maximum simulation time (t_max in seconds): 60

Output

Simulating Newton's Law of Cooling...

Time(s)	Temperature(C)
0.000000	80.000000
1.000000	76.850000
2.000000	73.972450
3.000000	71.350387
4.000000	68.967869

...

60.000000 25.021291